Applied Mathematical Sciences, Vol. 3, 2009, no. 12, 575 - 580

Characterization of Diagonalizable Matrices: An Algorithm

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Abstract. Let T be a linear opeartor and D be a matrix. So by its diagonal matrix, we get a lot of informations about T, namely we can almost answer any question about T. In this paper we introduce an efficient algorithm that characterizes whether a given matrix is diagonalizable in the field F or not (where F is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$).

Mathematics Subject Classification: 11C08, 65F30, 15A03, 15A06, 15A99

Keywords: Annihilator polynomial, diagonal matrix, linear operator, vector space

1. Introduction

For analysis of the linear operator T, we must know the annihilator polynomials class of T. That is, suppose V is a vector space over the field F and T is a linear operator on V. If p is a polynomial over F then $p(T)$ is an operator on V and if q is similar to p, then

$$(p+q)(T) = p(T) + q(T),$$
$$(pq)(T) = p(T)q(T).$$

Therefore the annihilator polynomials class of T, i.e., $p(T) = 0$ is an ideal in the polynomial algebra $F[x]$.
Note that if the field is of finite dimension then the ideal is nonzero. Since $F[x]$ is a PID, hence each polynomials ideal, consists of all coefficients, is a constant monic polynomial whose generating is an ideal. Accordingly the linear operator T with the monic polynomial p satisfies the following proposition.

Proposition 1.1. *If f is a polynomial over F then $f(T) = 0$ if and only if $f = pq$ where q is a polynomial over F.*

Proof. This is clear (see [3]). □

Definition 1.2. Suppose $T : V \to V$ is a linear operator on the vector space V of finite dimension over the field F. The minimal polynomial of T is the monic generating of the polynomials ideal over F that annihilate T.

Thus the minimal polynomial p_T has the following properties

1. p_T is a monic polynomial over the field F.
2. $p_T(T) = 0$.
3. Among all polynomials over F that annihilate T, p_T has the least degree.

Theorem 1.3. *Suppose that $T \in L(V)$, and (V, F) is a vector space of finite dimension over the field F. Hence*

1. *There exists the minimal polynomial p_T such that it is unique and its degree is at most $n = \dim(V)$.*
2. *If $q \in F[x]$ such that $q(T) = 0$ then there exists an another polynomial as $r \in F[x]$ such that $q = rp_T$.*

Now we present a straightforward way to the computation of minimal polynomial.
Suppose $v \in V$ and d is the least nonnegative integer such that the set $\{v, T(v), T_2(v), \dots , T_d(v)\}$ of vectors is linear dependent (obviously $d \le n$).
Note that $d = 0$ if and only if v is a zero vector and also $d = 1$ if and only if v is an eigenvector of T.
Accordingly there are the scalars $a_0, \dots , a_{d-1} \in F$ such that $T^d(v)+a_{d-1}T^{d-1}(v)+ \dots + a_1T(v) + a_0v = 0$. Now according to the property of d, we can assume that the coefficient $T^d(v)$ is 1. Accordingly

$$p_{T,v}(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1x + a_0 \in F[x].$$

Thus by Definition 1.2, $v \in \ker p_{T,v}(T)$. In other words, $p_{T,v}$ is a monic polynomial with the least degree such that $v \in \ker p_{T,v}(T)$.

Remark 1.4. if $q \in F[x]$ is a common divisor of p_{T,v_1} and p_{T,v_2} then v_1 and v_2 are in $\ker q(T)$.

We generalize the above method, if the set $B = \{v_1, v_2, \dots , v_n\}$ is a basis of V and q is a common divisor of $p_{T,v_1}, p_{T,v_2}, \dots , p_{T,v_n}$, then $B \subset \ker q(T)$, and therefore $q(T) = 0$. Hence we have the following theorem.

Theorem 1.5. *Suppose $T \in L(V)$, and (V, F) is a vector space of finite dimension over the field F. If the set $B = \{v_1, v_2, \dots , v_n\}$ is a basis of V, then p_T is the least common divisor of $p_{T,v_1}, p_{T,v_2}, \dots , p_{T,v_n}$.*

Proof. According to the above obtained results, $p(T) = 0$ and by Theorem 1.3, $p_T \mid p$. On the other hand

$$p_T = q_jp_{T,v_j} + r_j \text{ such that } \deg r_j < \deg p_{T,v_j} \quad (j = 1, 2, \dots , n).$$

Thus

$$0 = p_T(T)(v_j) = q_j(T)(p_{T,v_j}(T)v_j) + r_j(T)v_j = r_j(T)v_j.$$

We know that p_{T,v_j} has the least degree, hence $r_j = 0$, and since $p_{T,v_j} \mid p_T$ then $p_T = p$. □

We conclude that to the computation of the polynomials $p_{T,v_1}, p_{T,v_2}, \ldots, p_{T,v_n}$, we must find their least common divisor where $\{v_1, v_2, \ldots, v_n\}$ is a basis of the space (note that $\deg p_T \leq n$).

Example 1.6. We want to find the minimal polynomials for the following matrices

$$A = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Consider the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. Thus

$$Ae_1 = Be_1 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \quad A^2e_1 = B^2e_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad A^3e_1 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = Ae_1,$$

$$B^3e_1 = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} = -2B^2e_1 - Be_1 + 2e_1.$$

So we have

$$p_{A,e_1}(x) = x^3 - x \quad \text{and} \quad p_{B,e_1}(x) = x^3 + 2x^2 + x - 2.$$

Since $\deg p_{A,e_1} = 3$ and the minimal polynomial of A is a monic coefficient of p_{A,e_1} with at most degree 3, thus $p_A = p_{A,e_1}$ (without computation of p_{A,e_2} and p_{A,e_3}) and similarly $p_B = p_{B,e_1}$.

Theorem 1.7. *Suppose $T \in L(V)$ and (V, F) is a vector space of finite dimension over the field F. Then T is diagonalizable if and only if*

$$p_T(x) = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_k),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k \in F$ are distinct eigenvalues.

Proof. Follows from Theorem 1.5. □

2. Characterization of Diagonalizable Matrices: A Criterion

Our goal in this section is finding a subtle answer to the question, when can we write the polynomial $p \in F[x]$ in the form $p_T(x) = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_k)$?

First we must know two points, whether all roots of p have multiplicity 1, and whether these roots are belong to the field F.

We can answer the first question easily by the next theorem.

Theorem 2.1. *Suppose that $p \in F[x]$ is a nonconstant polynomial and p' is its derivation, then the following properties are equivalent*

1. *p has a root in F with the multiplicity greater than 1.*
2. *p and p' have a common root in F.*
3. *The greatest common divisor of p and p', i.e., $\gcd(p, p')$ has a root in F.*

Theorem 2.2. *Suppose $T \in L(V)$, and (V, F) is a vector space of finite dimension over the field F. Then*

1. *If F is an algebraically closed field (say $F = \mathbb{C}$), then T is diagonalizable if and only if $\gcd(p, p') = 1$.*
2. *If F is not an algebraically closed field (say $F = \mathbb{R}$), then T is diagonalizable if and only if $\gcd(p, p') = 1$ and all roots of p_T are in F.*

Proof. Follows from Theorem 1.7. □

Assume the finite sequence of real numbers $c = (c_0, \ldots, c_s) \in \mathbb{R}^{s+1}$. If $(c_0, \ldots, c_s)$ are nonzero then the number of variations in sign of c (V_c) is equal to the number of the indices $1 \leq j \leq s$ such that $c_{j-1}c_j < 0$. If some elements of c are zero, then the number of variations in sign of c (V_c) is equal to the number of variations in sign of the sequence consisting of nonzero elements of c.

Now suppose $p \in \mathbb{R}[x]$ is a nonconstant polynomial. The standard sequence corresponding to p is a sequence like $p_0, p_1, \ldots, p_s \in \mathbb{R}[x]$ such that

$$
\begin{aligned}
&p_0 = p \quad ; \quad p_1 = p' \\
&p_0 = q_1 p_1 - p_2 \quad ; \quad \deg p_2 < \deg p_1 \\
&\vdots \\
&p_{j-1} = q_j p_j - p_{j+1} \quad ; \quad \deg p_{j+1} < \deg p_j \\
&\vdots \\
&p_{s-1} = q_s p_s \quad ; \quad p_{s+1} \equiv 0.
\end{aligned}
$$

Now we are ready to see Sturm's Theorem.

Theorem 2.3 (Sturm's Theorem)**.** *Suppose $p \in \mathbb{R}[x]$ is a polynomial such that $\gcd(p, p') = 1$ and $a, b \in \mathbb{R}$, $a < b$ and $p(a)p(b) \neq 0$. Consider the standard sequence $p_0, p_1, \ldots, p_s \in \mathbb{R}[x]$ corresponding to p. Then the number of the roots of p in $[a, b]$ is $V_\alpha - V_\beta$ where $\alpha = (p_0(a), p_1(a), \ldots, p_s(a))$ and $\beta = (p_0(b), p_1(b), \ldots, p_s(b))$.*

Proof. See [4, §24, pp. 112-116] and [8]. □

Conclusion 2.4. *Suppose $p \in \mathbb{R}[x]$ is a nonconstant sequence such that $\gcd(p, p') = 1$ and the standard sequence corresponding to p is $p_0, p_1, \ldots, p_s \in \mathbb{R}[x]$ and $d_j = \deg p_j$ and c_j is leading coefficient of p_j ($j = 0, 1, \ldots, s$), then the number of the roots of p is $V_- - V_+$ where V_- is the number of variations in sign of the sequence $((-1)^{d_0}c_0, (-1)^{d_1}c_1, \ldots, (-1)^{d_s}c_s)$ and V_+ is the number of variations in sign of the sequence $(c_0, c_1, \ldots, c_s)$.*

3. Our Algorithm

If $T \in L(V)$ and (V, F) is a vector space of finite dimension over the field F where $F = \mathbb{R}$ or $F = \mathbb{C}$, then

step1. Compute the minimal polynomial p_T.

step2. Compute the standard sequence $p_0, p_1, \ldots, p_s$ corresponding to p.

- If p_s is not constant then T is not diagonalizable.
- If p_s is constant and $F = \mathbb{C}$ then T is diagonalizable.
- If p_s is constant anf $F = \mathbb{R}$ then go to step3.

step3. Compute V_- and V_+ corresponding to p_T. Hence T is diagonalizable if and only if

$$V_- - V_+ = \deg p_T.$$

Example 3.1. In Example 1.6, the minimal polynomial of the matrix B was $p_B(x) = x^3 + 2x^2 + x - 2$. The standard sequence corresponding to this polynomial is

$p_0(x) = x^3 + 2x^2 + x - 2$
$p_1(x) = x^2 + 4x + 1$
$p_2(x) = \frac{2}{9}x + \frac{20}{9}$
$p_3(x) = -261$

p_3 is constant, therefore B is diagonalizable. Now if $F = \mathbb{R}$, compute $V_- - V_+$, namely compute the number of variations in sign of the following sequences

$$(-1, 3, -\frac{2}{9}, -261) \quad \text{and} \quad (1, 3, \frac{2}{9}, -261).$$

Hence

$$V_- - V_+ = 2 - 1 = 1 < 3 = \deg(p_B).$$

Therefore, B is not diagonalizable over $\mathbb{R}$. On the other hand, the standard sequence corresponding to p_A is

$p_0(x) = x^3 - x$
$p_1(x) = 3x^2 - 1$
$p_2(x) = \frac{2}{3}x$
$p_3(x) = 1.$

Now clearly $V_+ = 0$ and $V_- = 3$. Therefore A is diagonalizable over $\mathbb{R}$. Furthermore, the matrices A and B are diagonalizable over $\mathbb{C}$, because their minimal polynomial is of degree 3, thus their eigenvalues are distinct.

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Received: April 30, 2008