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A Selective and Gradual Method for Efficiency Improvement of DEA Models

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Abstract

Basic-DEA models have ability to calculate performance targets for inefficient DMUs which are used for efficiency improvement approaches. Unfortunately, these improvement approaches aren't selective and gradual. Therefore, this paper introduces an efficiency improvement algorithm under this circumstance and discusses on the convergence and computational aspects of the algorithm. Also, an illustrative example is presented to show the ability of suggested method, from computational point of view.

Keywords: Data Envelopment Analysis, Efficiency Improvement algorithm, Elasticities

1. Introduction

Data Envelopment Analyses (DEA) is a linear programming based method which calculates relative efficiency of Decision Making Units (DMUs). It can include multiple outputs and inputs without recourse to a priori weights and without requiring explicit specification of functional forms between inputs and

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outputs. It computes a scalar measure of efficiency and determines efficient levels of inputs and outputs for each DMU.

Charnes et al. (1978-[6]) first proposed DEA as an evaluation tool to measure and compare a DMU's relative efficiency. Their model which is commonly referred to as a CCR model, assumed constant returns to scale. It was developed for variable returns to scale, by Banker et al. (1984-[3]). That is commonly referred to as a BCC model (for more detail see [5,7]).

Efficiency improvement is one of the important topics in DEA. literature, because, basic-DEA models have ability to calculate performance targets for inefficient DMUs which are used for efficiency improvement approaches. But, there are some difficulties for using efficiency improvement approaches which obtain by DEA models. For example, in DEA methodology the DMUs are supposed to be homogenous and comparable. This assumption is obviously not valid when there exist some extraordinary DMUs. In [2] examined the problems that arose due to some extraordinary DMUs and its influence on efficiency improvement of DMUs. Also, in the real world, it might not be possible to adjust all inputs and outputs of inefficient DMUs based on the DEA results; therefore, Kao [9] presented a modified version of DEA in which bounds are imposed on inputs and outputs. The results from his proposed model provide efficiency improvement for inefficient DMUs, which are feasible in practice.

A drawback of using basic-DEA models is its weakness in presenting selective efficiency improvement approaches. In other word, there isn't any choice for efficiency improvement level; hence, these improvement approaches aren't selective and gradual. Different methods have been introduced under this circumstance, which Inverse DEA models are the important group of them.[8,11], But, these methods make some assumption which are restrictive. Furthermore, to obtain results for each DMU, we should solve an IDEA model. This subject is sorely restrictive in real cases. Hence, a general and efficient method under this circumstance is necessary. Therefore the reminder of this paper is organized into 6 sections. Section 2 illustrates the proposed method as an efficiency improvement approach. Section 3 presents efficiency improvement algorithm. Section 4 discusses on the convergence and computational aspects of the algorithm. In Section 5, we present the ability our method by an illustrative example and after that, concluding remarks appear in section 6.

2. The Proposed Method

Consider observed output $Y_j = (y_{1j},\ldots,y_{sj}) \geq 0$ and input $X_j = (x_{1j},\ldots,x_{mj}) \geq 0$, $X_j \neq 0,\ Y_j \neq 0$ for *DMUj, j=1,...,n*. The DEA postulates that underlying the production possibility set (PPS) $T = \{(X,Y) |$ output vector $Y \geq 0$ can be produced from input vector $X \geq 0\}$ possess the following properties:

Postulate 1 (Nonempty). The observed $(X_j, Y_j) \in T$, *j=1,...,n*.

Postulate 2 (Proportionality). If $(X,Y) \in T$, then $(\lambda X, \lambda Y) \in T$ for all $\lambda \geq 0$.

Postulate 3 (convexity). T is a closed and convex set, i.e. if $(X_1, Y_1) \in T$ and $(X_2, Y_2) \in T$ then for $\lambda \in (0,1),\ \lambda(X_1, Y_1) + (1-\lambda)(X_2, Y_2) \in T.$

Postulate 4 (Plausibility). If $(X, Y) \in T, X_t \geq X, Y_t \leq Y$, then $(X_t, Y_t) \in T$.

Postulate 5 (Minimum extrapolation). T is the smallest set satisfies postulates 1-4.

The above-mentioned postulates define the following unique set:

$$T_c = \left\{ (X_t, T_t) \,|\, X_t \geq \sum_{j=1}^{n} \lambda_j X_j \ \ , Y_t \leq \sum_{j=1}^{n} \lambda_j Y_j ,\ \lambda_j \geq 0,\ j = 1,\ldots,n \right\}.$$

For assessing the relative efficiency of DMUp which is defined from PPS we have the following problem:

$$\begin{aligned} & Min\ \theta_{p,} \\ & s.t.\ \ (\theta_p X_p, Y_p) \in T_c . \end{aligned}$$

By considering T_c we have

$$\begin{aligned} & Min\ \theta_{p,} \\ & s.t.\ \ \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_p x_{ip}, \qquad i = 1,\ldots,m, \\ & \qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \quad r = 1,\ldots,s, \\ & \qquad \lambda_j \geq 0, \quad j = 1,\ldots,n. \end{aligned}$$

This model is CCR input-oriented model; similarly CCR output-oriented model can be defined.

The dual problem of above model can be formulated as follows:

$$\begin{aligned} & Max \quad \sum_{r=1}^{s} u_r y_{rp} \\ & s.t. \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j = 1,\ldots,n \\ & \qquad \sum_{i=1}^{m} v_i x_{ip} = 1 \\ & \qquad u, v \geq \varepsilon \end{aligned}$$

Where, the variables are u_r and v_i as

u_r : weight assigned to output *r* (*r=1,…,s*),

v_i : weight assigned to input *i* (*i=1,…,m*).

Also the non-Archimedean infinitesimal Epsilon is used in the model for some computational considerations, for more details see [1,10]. But, this model is equivalent with a fractional programming problem which is shown in below (see [4]):

$$EFF_p = Max \quad \sum_{r=1}^{s} u_r y_{rp} / \sum_{i=1}^{m} v_i x_{ip}$$

$$s.t. \quad \sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij} \leq 1 \qquad j = 1,...,n$$

$$u_r, v_i \geq 0$$

This model is CCR multiplier model, But, $\sum_{r=1}^{s} u_r y_{rj}$ and $\sum_{i=1}^{m} v_i x_{ij}$ are total revenue and total cost for *j*th DMU, respectively, which are calculated in an optimization problem. Therefore, EFF_p shows the relationship between total revenue and total cost in *p*th DMU:

$$Total\ Revenue(TR_p) = Eff_p \times Total\ Cost(TC_p)$$

Definition: *The elasticity of input in the output*.

Relative change of an output quantity by one percent change of an input is the elasticity of that input in that output:

$$E_{ij} = \frac{\delta Y_i}{\delta X_j} \cdot \frac{X_j}{Y_i}$$

Where, E_{ij} is the elasticity quantity of input *j* in the output *i*.

Theorem. The elasticity of input *i* in the total revenue and the elasticity of output *j* in the total cost, respectively, in the CCR model is:

$$ex_{ip} = \frac{v_{ip} x_{ip}}{\sum_i v_{ip} x_{ip}} \qquad where \qquad \sum_i ex_{ip} = 1 \qquad (I)$$

$$ey_{jp} = \frac{u_{jp} y_{jp}}{\sum_j u_{jp} y_{jp}} \qquad where \qquad \sum_j ey_{jp} = 1 \qquad (II)$$

Proof. Suppose, there are *n* decision making units. Output vector and input vector for the *p*th DMU are $Y_p = (y_{1p},..., y_{sp})$ and $X_p = (x_{1p},...,x_{mp})$, respectively. We calculate CCR model (multiplier model) for measuring efficiency of *p*th DMU, so objective function for this DMU is:

$$Eff_p = \frac{Total\ Revenue\ (TR_p)}{Total\ Cost\ (TC_p)} = \frac{\sum_j u_{jp} y_{jp}}{\sum_i v_{ip} x_{ip}}$$

or

$$TR_p = Eff_p \times TC_p$$

Therefore, as noted in previous definition, the elasticity of input *i* in the TR_p is calculated as follow:

$$\begin{cases} TR_p = Eff_p \times \sum_i v_{ip} x_{ip} \\ \dfrac{\delta TR_p}{\delta x_{ip}} = Eff_p \times v_{ip} \end{cases}$$

$$ex_{ip} = \frac{\delta TR_p}{\delta x_{ip}} \times \frac{x_{ip}}{TR_p} = Eff_p \times v_{ip} \times \frac{x_{ip}}{Eff_p \times \sum_i v_{ip} x_{ip}} = \frac{v_{ip} x_{ip}}{\sum_i v_{ip} x_{ip}}$$

$$ex_{ip} = \frac{v_{ip} x_{ip}}{\sum_i v_{ip} x_{ip}} \quad where \quad \sum_i ex_{ip} = 1 \qquad (I)$$

Similarly, the elasticity of output *j* in the TC_p is calculated:

$$ey_{jp} = \frac{u_{jp} y_{jp}}{\sum_j u_{jp} y_{jp}} \quad where \quad \sum_j ey_{jp} = 1 \qquad (II).•$$

Therefore,

• relationship between the quantities of *i_th* input elasticities and efficiency for each DMU is:

$$ex_{ip} = \frac{\Delta Eff}{\Delta x_i} \times \frac{x_i}{Eff} \quad \Leftrightarrow \quad \Delta x_i = \frac{x_i \times \Delta Eff}{Eff \times ex_i} \qquad (EX$$

Similarly,

• relationship between the quantities of *j_th* output elasticities and efficiency for each DMU is:

$$ey_{jp} = \frac{\Delta Eff}{\Delta y_j} \times \frac{y_j}{Eff} \quad \Leftrightarrow \quad \Delta y_j = \frac{y_j \times \Delta Eff}{Eff \times ey_{jp}} \qquad (EY$$

In addition, using elasticities which are obtained by DEA models (CCR model here), we can obtain effective factors on efficiency changes for each DMU. Hence, above formulas present an efficiency improvement plan for each DMU privately. In other word, using elasticities, we can choose logical efficiency improvement for each DMU and obtain new inputs (outputs) to achieve the quantity of selective efficiency.

Now to implement the efficiency improvement process, we use the following algorithm: (Notice, this algorithm is for CCR model and it can be extended for other models)

3. Efficiency Improvement Algorithm

1. Let S be the set of all DMU's and $i \leftarrow 1$.
2. Run the CCR input-oriented model for all DMUs in S.
3. Calculate the output and input elasticities for all DMUs in S.

4. Select DMU j and decide for the quantity of efficiency improvement and put it in ΔEff_j.
5. Select orientation of changing for efficiency improvement (output orientation or input orientation)
6. If output orientation has been selected go to 10, otherwise continue.
7. Proportion the quantity of ΔEff_j between output elasticities.
8. Calculate outputs changes by EY formula.
9. Add the quantities of outputs changes with old outputs, respectively, next go to 13.
10. Proportion the quantity of ΔEff_j between input elasticities.
11. Calculate input changes by EX formula.
12. Subtract the quantities of inputs changes of old input, respectively.
13. In this stage the efficiency improvement solutions for DMU j is produced. For the next DMU the algorithm will be repeated, therefore $i \leftarrow i+1$ and go to 4 otherwise this process is finished.

4. Algorithm Convergence and its Computational Aspects

This fact that the number of the DMU's is finite, so the algorithm convergence is guarantee.

From the computational point of view this algorithm is divided into two basic phases. The first phase, calculates efficiency scores and elasticities for all of DMUs and the second phase manages the process of performance analysis and calculate efficiency improvement solutions. The main computation effort is in the first phase when we run model for all decision making units. In the second phase of the algorithm we deal with small computation that is not time consuming which is related to calculation of efficiency improvement solutions.

5. An Illustrative Example

In this section we illustrative our method by an example. Therefore, consider table 1. In this table we have seven DMUs with two inputs and three outputs. We employ CCR input-orientation for calculating efficiency score and elasticities for all of DMUs.

Table 1, Data of seven DMUs with two inputs and three outputs.

DMU	Input 1	Input 2	Output 1	Output 2	Output 3
1	300	4000	300	1.11	2
2	400	7000	200	3.33	7
3	200	2000	175	1.25	5
4	150	6000	200	1.42	3
5	350	3000	125	3.50	9
6	200	600	60	2.00	3
7	80	100	30	5.00	5

Table 2, presents our results of input and output elasticities for each DMU. Also efficiency score for each DMU is given in this table.

Table 2, result of elasticities and efficiency score for each DMU.

DMU	Efficiency	E-Input 1	E-Input 2	E-Output 1	E-Output 2	E-Output 3
1	1.00	0.54	0.46	0.99	0.01	0.00
2	0.55	0.80	0.20	0.75	0.00	0.25
3	1.00	0.88	0.12	0.78	0.00	0.23
4	1.00	0.65	0.35	0.87	0.00	0.13
5	0.56	0.90	0.10	0.58	0.00	0.42
6	0.63	0.64	0.36	1.00	0.00	0.00
7	1.00	0.81	0.19	1.00	0.00	0.00

For example, efficiency score for DMU 2 is *0.55*, also, input elasticities for this DMU are *0.80* and *0.20* respectively, and output elasticities are *0.75*, *0.00* and *0.25*. According to the elasticities, input 1 among all inputs and output 1 among all outputs are effective factors on efficiency changes for current DMU. This information is useful to make strategic map for performance improvement by efficiency improvement.

For instance, we try to find out the efficiency improvement plan for DMU 2. Therefore, consider table 3 which show conditions for each improvement subject.

Table 3: three subject for efficiency improvement of DMU 2.

	Efficiency improvement	Orientation of Changing
Subject 1	0.05	Inputs
Subject 2	0.45	Inputs
Subject 3	0.10	Outputs

Subject 1: in this subject, the purpose is detecting new inputs, according to the 0.05 change in efficiency. But input elasticities are $ex_1 = 0.80$ and $ex_2 = 0.20$, therefore, proportional efficiency change for each input is:

$$\Delta Eff = 0.05 \quad \begin{array}{l} \rightarrow \Delta Eff_1 = 0.04 \\ \rightarrow \Delta Eff_2 = 0.01 \end{array}$$

Then, changes in inputs are calculated:

$$\Delta x_i = \frac{x_i \times \Delta Eff}{Eff \times ex_i} \quad \begin{array}{l} \rightarrow \Delta x_1 = \dfrac{400 \times 0.04}{0.50 \times ex_1} = 36.36 \\ \rightarrow \Delta x_2 = \dfrac{7000 \times 0.01}{0.50 \times ex_2} = 636.36 \end{array}$$

And new inputs are calculated by subtracting inputs changes of old input:

$\rightarrow\ x_1\,(New) = 400 - 36.36 = 363.64$

$\rightarrow\ x_2\,(New) = 7000 - 636.36 = 6363.64$

Therefore, with new inputs, efficiency score for current DMU become *0.60.*

Subject 2: $\Delta Eff = 0.45$ and Change in inputs**.**

in this subject, the purpose is detecting new inputs, according to the 0.45 change in efficiency, therefore, inputs changes and new inputs are:

$$\Delta Eff = 0.45 \quad \begin{cases} \rightarrow\ \Delta Eff_1 = 0.36\ \Rightarrow\ \Delta x_1 = 327.27\ \Rightarrow\ x_1\,(New) = 400 - 327.27 = 72.73 \\ \rightarrow\ \Delta Eff_2 = 0.09\ \Rightarrow\ \Delta x_2 = 5727.27\ \Rightarrow\ x_2\,(New) = 7000 - 5727.27 = 1272.73 \end{cases}$$

Therefore, with these new inputs, efficiency score for current DMU become *1.00;* hence, it will be efficient.

Subject 3: in this subject, the purpose is detecting new outputs, according to the *0.10* change in efficiency. But output elasticities are $ey_1 = 0.75$, $ey_2 = 0$ and $ey_3 = 0.25$, therefore, proportional efficiency change for each output is:

$$\Delta Eff = 0.10 \quad \begin{cases} \rightarrow\ \Delta Eff_1 = 0.075 \\ \rightarrow\ \Delta Eff_1 = 0.00 \\ \rightarrow\ \Delta Eff_2 = 0.025 \end{cases}$$

Then, changes in outputs are calculated:

$$\Delta y_j = \frac{y_j \times \Delta Eff}{Eff \times ey_j} \quad \begin{cases} \rightarrow\ \Delta y_1 = \dfrac{200 \times 0.075}{0.50 \times 0.75} = 36.36 \\ \rightarrow\ \Delta y_2 = 0 \\ \rightarrow\ \Delta y_3 = \dfrac{7 \times 0.025}{0.50 \times 0.25} = 1.2727 \end{cases}$$

And new outputs are calculated by adding the quantities of outputs changes with old outputs:

$\rightarrow\ y_1\,(New) = 200 + 36.36 = 236.36$

$\rightarrow\ y_2\,(New) = 3.33 + 0 = 3.33$

$\rightarrow\ y_3\,(New) = 7 + 1.2727 = 8.2727$

Therefore, with these new outputs, efficiency score for current DMU become *0.60.*

6. Conclusion

In this paper, efficiency improvement algorithm was introduced. This method, presents efficiency improvement approaches which can be selective and gradual. Using elasticities which are obtained by DEA models, it presents effective factors on efficiency changes for each DMU, therefore, it is able to present efficiency improvement plan for each DMU privately which is given as an algorithm in this paper. Also, the convergence and computational aspects of the algorithm were discussed. An illustrative example was presented to show the ability of suggested algorithm, from computational point of view.

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