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Stochastic Monoids

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Abstract

We introduce stochastic monoids and stochastic congruences and we investigate their basic properties.

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Let M be a non empty set and $[0,1]$ the unit interval. A function $f : M \to [0,1]$ is said to be *stochastic* if the sum of its values $\sum_{m\in M} f(m)$ exists and is equal to 1. We denote by $STOCH(M)$ the so defined set. Next strong Convexity Lemma is useful.

Lemma 1. *Given a family of stochastic functions $f_i \in STOCH(M)$, $i \in I$ and a stochastic family $(\lambda_i)_{i\in I}$ of numbers in $[0,1]$, then the function $\sum_{i\in I} \lambda_i f_i$ exists and is stochastic as well.*

Hence, for any stochastic function $f : M \to [0,1]$ we have the expansion formula

$$f = \sum_{m\in M} f(m)\hat{m}$$

where $\hat{m} : M \to [0,1]$ stands for the singleton function $\hat{m}(m) = 1$, $\hat{m}(n) = 0$ for $n \neq m$. Often $\hat{m}$ is identified with m itself.

A stochastic monoid is a set M equipped with a stochastic multiplication, i.e. a function

$$M \times M \to STOCH(M), \quad (m_1, m_2) \mapsto m_1 m_2$$

which is associative

$$\sum_{n \in M} m_1(n)(m_2 m_3)(n) = \sum_{m \in M} (m_1 m_2)(n) m_3(n)$$

and unitary i.e. there is an element $e \in M$ such that

$$me = m = em, \text{ for all } m \in M.$$

For instance any ordinary monoid can be viewed as a stochastic monoid. In the present study it is important to have a congruence notion. More precisely, let M be a stochastic monoid and $\sim$ an equivalence relation on the set M, such that $m_1 \sim m_1'$ and $m_2 \sim m_2'$ implies

$$\sum_{m \in [n]} (m_1 m_2)(n) = \sum_{m \in [n]} (m_1' m_2')(n)$$

for all $\sim$-classes $[m]$, $m \in M$. Then $\sim$ is called a *stochastic congruence* on M.

The quotient set $M/\sim$ is then structured into a stochastic monoid by defining the stochastic multiplication via the formula

$$([m_1][m_2])([m]) = \sum_{n \in [m]} (m_1 m_2)(n)$$

Congruences on an ordinary monoid M coincide with stochastic congruences when M is viewed as a stochastic monoid. The first question arisen is whether stochastic congruence is a good algebraic notion. This is checked by the validity of the known isomorphism theorems in their stochastic variant.

Given stochastic monoids M and N, any function $h : M \to N$ preserving stochastic multiplication and units

$$\bar{h}(m_1 m_2) = h(m_1) h(m_2), \quad h(e) = e', \quad \text{for all } m_1, m_2 \in M,$$

is called a *morphism* from M to N.

The above function $\bar{h} : STOCH(M) \to STOCH(N)$ is the stochastic extension of h:

$$\bar{h} : (f) = \sum_{m \in M} f(m)h(m)$$

which exists because of the strong convexity lemma.

Theorem 1. *Given an epimorphism $h : M \to N$ and a stochastic congruence $\sim$ on N, its inverse image $h^{-1}(\sim)$ defined by*

$$m_1 \equiv m_2 h^{-1}(\sim) \;\; if \;\; h(m_1) \sim h(m_2)$$

is also a stochastic congruence and the stochastic quotient monoids $M/h^{-1}(\sim)$ and $N/\sim$ are isomorphic.

Given stochastic monoids $M_1, \ldots, M_k$ the stochastic multiplication

$$(m_1, \ldots, m_k) \cdot (m'_1, \ldots, m'_k)$$

$$= \sum_{n_1 \in M_1, \ldots, n_k \in M_k} (m_1 m'_1)(n_1) \cdots (m_k m'_k)(n_k)(n_1, \ldots, n_k)$$

structures the set $M_1 \times \cdots \times M_k$ into a stochastic monoid so that the canonical projection

$$\pi_i : M_1 \times \cdots \times M_k \to M_i, \quad \pi_i(m_1, \ldots, m_k) = m_i$$

becomes a morphism of stochastic monoids.

Theorem 2. *Let $\sim_i$ be a stochastic congruence on the stochastic monoid M_i $(1 \leq i \leq k)$. Then $\sim_1 \times \cdots \times \sim_k$ is a stochastic congruence on the stochastic monoid $M_1 \times \cdots \times M_k$ and the stochastic monoids $M_1 \times \cdots \times M_k / \sim_1 \times \cdots \times \sim_k$ and $M_1/\sim_1 \times \cdots \times M_k/\sim_k$ are isomorphic*

Congruences on fuzzy algebras have been studied in [LB].

References

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