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On a Truncated Erlangian Queueing System with State – Dependent Service Rate, Balking and Reneging

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Abstract

The aim of this paper is to derive the analytical solution of the truncated Erlangian service queue with state-dependent rate, balking and reneging (M/ER/I/N (α, β)) . We obtain $P_{n,s}$, the probabilities that there are "n" units in the system and the unit in the service occupces stage "s" (s = 1, 2,…, r) .We treat this queue for general values of r , k and N.

Keywords: truncated Erlangian service queue, balking, reneging.

1 INTRODUCTION

This paper considers the queueing system M/Er/1/N with state – dependent service rate, balking and reneging concepts .The Erlang distrbution, denoted by Er is a special case of the gamma distribution, is named after A.K. Erlang who pioneered queueing system theory for its application to congestion in telphone networks .The nontruncated queue: M/Er/1 was solved by Morse [3] at r =2 and white et al. [4] Who obtained the solution in the form of a generating function and the probabilities could be obtained by a power series expansion. Al Seedy [1] gave an analytical solution of the queue:

M/Er/1/N with balking only. This work had been followed by Kotb [2] who studied the analytical solution of the state – dependent Erlangian queue: M/Er/1/N with balking by using a very useful lemma. In this paper we treat the analytical solution of the queue: M/Er/1/N (α,β) for finite capacity considering by using a recurrence relations .We obtain $P_{n,s}$, the probabilities that there are "n" units in the system and the unit in serive occupies stage "s" ($1 \le s \le r$ in terms of P_0 .

The probability of an empty system P_0 is also obtained .The discipline considered is first in first out (FIFO).

2 THE PROBLEM ANALYSIS

Consider the single – channel service time Erlangian queue having r – service stages each with rate μ_n, with the state – dependent and reneging in the form:

$$\mu_K = \begin{cases} r\mu_1 & n = 1, \\ r\mu_1 + (n+1)\alpha, & 2 \le n \le k,\ \mu_1 < \mu_2, \\ r\mu_2 + (n-1)\alpha, & k+1 \le n \le N, \end{cases}$$

where α is the rate of time t, having the (P.$d.f$) given by :

$$f(t) = \alpha e^{-\alpha t}, \qquad t \ge o\ ,\ \alpha > 0.$$

This means that the units are served with two different rates $r\mu_1$ or $r\mu_2$ depending on the number of units in the system whether $1 \le n \le k$ or $k+1 \le n \le N$ respectively.

Also, consider an exponential interarrival pattern with rate λ_n. Assume $(1-\beta)$ be the probability that a unit balks (does not enter the queue).

where : β= p(a unit joins the queue), $0 \le \beta < 1$, $1 \le n \le N$;

For $\beta = 1$, $n = 0$, it is clear that:

$$\lambda_n = \begin{cases} \lambda & n = 0 \\ \beta\lambda & 1 \le n \le N \end{cases}$$

Assume the probabilities**:**

$P_{n,s}$ = p (n units in the system and the unit in service being in stage s),

where: $1 \le n \le N$, $1 \le s \le r$

P_o = probability of an empty system, i.e. the dally probability. The steady – state difference equations are:

$$\lambda P_o - r\mu_1 P_{1,1} = 0\,, \qquad n=0 \tag{1}$$

$$\left.\begin{array}{ll} (r\mu_1 + \beta\lambda)P_{1,s} - r\mu_1 P_{1,s+1} = 0\,, & 1 \le s \le r-1 \\ (r\mu_1 + \beta\lambda)P_{1,r} - \lambda P_o - (r\mu_1 + \alpha)P_{2,1} = 0\,, & s=r \end{array}\right\} \quad n=1 \tag{2}$$

$$\left.\begin{array}{ll} (r\mu_1 + (n-1)\alpha + \beta\lambda)P_{n,s} - (r\mu_1 + (n-1)\alpha)P_{n,s+1} = 0, & 1 \le s \le r-1 \\ (r\mu_1 + (n-1)\alpha + \beta\lambda)P_{n,r} - \beta\lambda P_{n-1,r} - (r\mu_1 + (n-1)\alpha)P_{n+1,1} = 0, & s=r \end{array}\right\},\ 2 \le n \le k-1 \tag{3}$$

$$\left.\begin{array}{ll} (r\mu_1 + (k-1)\alpha + \beta\lambda)P_{k,s} - \beta\lambda P_{k-1,s} - (r\mu_1 + (k-1)\alpha)P_{k,s+1} = 0, & 1 \le s \le r-1 \\ (r\mu_1 + (k-1)\alpha + \beta\lambda)P_{k,r} - \beta\lambda P_{k-1,r} - (r\mu_2 + k\alpha)P_{k,1+1} = 0, & s=r \end{array}\right\},\ n=k \tag{4}$$

$$\left.\begin{array}{ll} (r\mu_2 + (n-1)\alpha + \beta)P_{n,s} - \beta\lambda P_{n-1,s} - (r\mu_2 + (n-1)\alpha)P_{n,s+1} = 0, & 1 \le s \le r-1 \\ (r\mu_2 + (n-1)\alpha + \beta)P_{n,r} - \beta\lambda P_{n-1,r} - (r\mu_2 + n\alpha)P_{n,1+1} = 0, & s=r \end{array}\right\},\ k+1 \le n \le N-1 \tag{5}$$

$$\left.\begin{array}{ll} (r\mu_2 + (N-1)\alpha)P_{n,s} - \beta\lambda P_{n-1,s} - (r\mu_2 + (N-1)\alpha)P_{n,s+1} = 0 & ,1 \le s \le r-1 \\ (r\mu_2 + (N-1)\alpha)P_{n,r} - \beta\lambda P_{n-1,r} = 0, & s=r \end{array}\right\},\ n = N \tag{6}$$

Summing (2) over s and using (1), gives

$$P_{2,1} = \frac{\beta\lambda}{(r\mu_1 + \alpha)} \sum_{s=1}^{r} P_{1,s}\,, \qquad n=2 \tag{7}$$

Summing (3) over s, using (7) and adding the results obtaining for $2 \le n \le k-1$, leads to:

$$P_{n,1} = \frac{\beta\lambda}{r\mu_1 + (n-1)\alpha} \sum_{s=1}^{r} P_{n-1,s}\,, \quad 3 \le n \le k \tag{8}$$

Similarly, summing (4) over s, and using (8) at n=k, yields

$$P_{k+1,1} = \frac{\beta\lambda}{(r\mu_2 + k\alpha)} \sum_{s=1}^{r} P_{k,s}\,, \qquad n = k+1 \tag{9}$$

Summing (5) over s, and using (9):

$$P_{n,1} = \frac{\beta\lambda}{(r\mu_2 + (n-1)\alpha)} \sum_{s=1}^{r} P_{n-1,s} , \qquad k+2 \le n \le N \tag{10}$$

From equation one can easily show that

$$P_{1,1} = \phi_1 P_0$$

Making use of equation (2), yields

$$P_{1,s} = \phi_1 (1+\beta\phi_1)^{s-1} P_0 , \qquad 1 \le s \le r \tag{11}$$

Upon using the first equation of (3) and (8) we get the recurrence relation.

$$P_{n,s} = \beta\phi_n (1+\beta\phi_n)^{s-1} \left\{ \sum_{i=1}^{r} P_{n-1,i} - \sum_{i=1}^{s-1} \left(\frac{1}{1+\beta\phi_n} \right)^{i} P_{n-1,i} \right\} , \qquad 2 \le n \le k \tag{12}$$

Also, from the first Equation of (5) and (10), we obtain

$$P_{n,s} = \beta\phi_n (1+\beta\phi_n)^{s-1} \left\{ \sum_{i=1}^{r} P_{n-1,i} - \sum_{i=1}^{s-1} \left(\frac{1}{1+\beta\phi_n} \right)^{i} P_{n-1,i} \right\} , \quad k+1 \le n \le N-1 \tag{13}$$

Finally, using equation (6) and equation (10) at n=N, gives:

$$P_{N,s} = \beta\phi_N \sum_{i=s}^{r} P_{N-1,i} , \quad 1 \le s \le r, \tag{14}$$

where:

$$\phi_n = \begin{cases} \dfrac{\lambda}{r\mu_1 + (n-1)\alpha} , & 1 \le n \le k \\ \dfrac{\lambda}{r\mu_2 + (n-1)\alpha} , & k+1 \le n \le N \end{cases}$$

Equations (11) – (14) are the required recurrence relations, that give all probabilities in terms of P_0 which it-self may now be determined by using the normalizing condition:

$$P_0 + \sum_{n=1}^{n} \sum_{s=1}^{r} P_{n,s} = 1, \tag{15}$$

Hence all the probabilities are completely known in terms of the queue parameters.

3 EXAMPLE:

The following example illustrates the theoretical results. In the system: $M/E_r/1/N$ with state-dependent , balking and reneging , let k =2, r =3 and N= 4, (i.e, the queue $M/E_3/1/4$ (α, β)) , in the equations (11) – (15) , the results are :

$$P_{1,1}=a_1 P_0, \quad P_{1,2}=a_2 P_0, \quad P_{1,3}=a_3 P_0.,$$
$$P_{2,1}=b_1 P_0, \quad P_{2,2}=b_2 P_0, \quad P_{2,3}=b_3 P_0.,$$
$$P_{3,1}=c_1 P_0, \quad P_{3,2}=c_2 P_0, \quad P_{3,3}=c_3 P_0.,$$
$$P_{4,1}=d_1 P_0, \quad P_{4,2}=d_2 P_0, \quad P_{4,3}=d_3 P_0.,$$

where:

$$a_1=\phi_1, a_2=\phi_1\left(1+\beta_{\ 1}\phi\right), a_3=\phi_1\left(1+\beta\phi_1\right)^2$$
$$b_1=\beta\phi_2\left(a_1+a_2+a_3\right),$$
$$b_2=\beta\phi_2\left(1+\beta\phi_2\right)\left\{a_1+a_2+a_3-\left(\frac{1}{1+\beta\phi_2}\right)a_1\right\},$$
$$b_3=\beta\phi_2\left(1+\beta\phi_2\right)^2\left\{a_1+a_2+a_3-\left(\frac{1}{1+\beta\phi_2}\right)a_1-\left(\frac{1}{1+\beta\phi_2}\right)^2 a_2\right\},$$
$$c_1=\beta\phi_3\left(b_1+b_2+b_3\right) \quad ,$$
$$c_2=\beta\phi_3\left(1+\beta\phi_3\right)\left\{b_1+b_2+b_3-\left(\frac{1}{1+\beta\phi_3}\right)b_1\right\} \quad ,$$
$$c_3=\beta\phi_3\left(1+\beta\phi_3\right)^2\left\{b_1+b_2+b_3-\left(\frac{1}{1+\beta\phi_3}\right)b_1-\left(\frac{1}{1+\beta\phi_3}\right)^2 b_2\right\},$$

$$d_1=\beta\phi_4\left(c_1+c_2+c_3\right),$$
$$d_2=\beta\phi_4\left(c_2+c_3\right),$$
$$d_3=\beta\phi_4 c_3,$$

$$\phi_1=\frac{\lambda}{3\mu_1}, \qquad \phi_2=\frac{\lambda}{3\mu_1+\alpha},$$
$$\phi_3=\frac{\lambda}{3\mu_2+2\alpha}, \qquad \phi_4=\frac{\lambda}{3\mu_2+3\alpha}.$$

From the normalizing condition:

$$P_0+\sum_{s=1}^{3}P_{1,s}+\sum_{s=1}^{3}P_{2,s}+\sum_{s=1}^{3}P_{3,s}+\sum_{s=1}^{3}P_{4,s}=1,$$

we have

$$P_0 = \{1 + a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3 + d_1 + d_2 + d_3\}^{-1}.$$

Therefore, the expected numbers in the system and in the queue are, respectively.

$$L = \sum_{n=1}^{4}\sum_{s=1}^{3} n P_{n,s}$$

$$= \{a_1 + a_2 + a_3 + 2(b_1 + b_2 + b_3) + 3(c_1 + c_2 + c_3) + 4(d_1 + d_2 + d_3)\}P_0$$

$$Lq = \sum_{n=1}^{4}\sum_{s=1}^{3} (n-1) P_{n,s}$$

$$= \{(b_1 + b_2 + b_3) + 2(c_1 + c_2 + c_3) + 3(d_1 + d_2 + d_3)\}P_0 .$$

Also the expected waiting time in Kotb the system and the queue are obtained as follows:

$$W = \frac{L}{\lambda'} , \quad Wq = \frac{lq}{\lambda'} , \lambda' \left(L - L_q\right)\mu , \quad \mu = \frac{1}{2}(\mu_1 + \mu_2) \ .,$$

where λ' is the mean rate of units actually entering the system .

4 SPECIAL CASES:

CASE 1: Let $\alpha = 0$ *and* $k \to \infty \left(\mu_1 = \mu_2 = \mu \ , k = N\right).$

Our results agree with the results of Al-Seedy [1].

CASE 2: Results of both (2), has been obtained by letting $\alpha = o$ in the equations (11) – (14) in our results.

CASE 3: Results of (4), can be obtained by letting K, N $\to \infty, \alpha = o$ and $\beta = 1$ in our results.

5 CONCLUSION:

In this paper, the truncated Erlangian service queue is studied with state – dependent, balking and reneging .The recurrence relations that gave all the probabilities in terms of P_0 are derived. We illustrate the method by an example is give to obtain some measures of effectiveness such as L, Lq, w, w_q.

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