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On Hilbert's Integral Inequality and Applications

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Abstract

In this paper it is shown that a new refinement on Hilbert's integral inequality can be established by introducing a weight function of the form $\frac{1}{2}\left(\cos 2\sqrt{x-\alpha} - e^{-2\sqrt{x-\alpha}}\right)$ (with $x-\alpha \geq 0$). As application, some sharp results of Widder's inequality and Hardy-Littlewood's inequality are obtained.

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1 Introduction and Lemmas

Let $f(x),\ g(x) \in L^2(0, +\infty)$. It is all known that the inequality of the form

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \leq \pi \left\{ \int_0^\infty f^2(x)\,dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{1/2}. \tag{1.1}$$

Is called Hilbert's integral inequality, where the coefficient π is the best possible.

Inequality (1.1) has the following extension of the form

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)\,g(y)}{x+y-2\alpha}dxdy \le \pi\left\{\int_{\alpha}^{\infty} f^2(x)\,dx\right\}^{1/2}\left\{\int_{\alpha}^{\infty} g^2(x)\,dx\right\}^{1/2}. \tag{1.2}$$

where $x-\alpha \ge 0$ and $y-\alpha \ge 0$.

Recently, various improvements and extensions of (1.1) and (1.2) appear in a great deal of papers (see [1]). In this paper we will give some new improvements of (1.1) and (1.2), and the method adopted by us has trait itself, it is different from those listed in the paper [1]. Explicitly, the idea and the results obtained possess new meanings.

In order to prove our assertion, we need the following lemmas.

Lemma 1.1. Let $c(x)$ be an integrable function in the interval $(0,+\infty)$,

$$J_1=\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f^2(x)}{x+y-2\alpha}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{1}{2}}\left(1-c(x-\alpha)+c(y-\alpha)\right)dxdy$$

$$\text{and}\quad J_2=\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f^2(y)}{x+y-2\alpha}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{1}{2}}\left(1-c(x-\alpha)+c(y-\alpha)\right)dxdy \tag{1.3}$$

where $x-\alpha \ge 0$ and $y-\alpha \ge 0$. Then

$$J_1J_2=\pi^2\left\{\left(\int_{\alpha}^{\infty} f^2(x)\,dx\right)^2-\left(\int_{\alpha}^{\infty} k(x)\,f^2(x)\,dx\right)^2\right\} \tag{1.4}$$

$$\textit{where}\quad k(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{c((x-\alpha)t^2)}{1+\ t^2}dt-c(x-\alpha) \tag{1.5}$$

$$proof.\ J_1 = \int_\alpha^\infty \int_\alpha^\infty \frac{f^2(x)}{(x-\alpha)\left(1-\frac{y-\alpha}{x-\alpha}\right)} \left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{1}{2}} (1 - c(x-\alpha) + c(y-\alpha))\, dxdy.$$

$$= \int_\alpha^\infty \left\{ \int_\alpha^\infty \frac{1}{(x-\alpha)(1-\frac{y-\alpha}{x-\alpha})} \left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{1}{2}} (1 - c(x-\alpha) + c(y-\alpha))\, dy \right\} f^2(x)\, dx$$

$$= \int_\alpha^\infty \left\{ \pi + \int_0^\infty \frac{c((x-\alpha)u)}{1+u} \left(\frac{1}{u}\right)^{\frac{1}{2}} du - \pi c(x-\alpha) \right\} f^2(x)\, dx$$

$$= \int_\alpha^\infty \left\{ \pi + 2\int_0^\infty \frac{c\left((x-\alpha)\, t^2\right)}{1+\ t^2} dt - \pi c(x-\alpha) \right\} f^2(x)\, dx$$

$$= \pi \left(\int_\alpha^\infty f^2(x)\, dx + \int_\alpha^\infty k(x)\, f^2(x)\, dx \right)$$

where $k(x)$ is a function defined by (1.5).

Similarly, we have

$$J_2 = \pi \left(\int_\alpha^\infty f^2(x)\, dx - \int_\alpha^\infty k(x)\, f^2(x)\, dx \right)$$

It follows that the relation (1.4) holds.

Lemma 1.2. If $c(x) = \sin^2 \sqrt{x}$, $x \in (0,\ +\infty)$, then

$$\int_0^\infty \frac{c(xt^2)}{1\ +\ t^2} dt = \frac{\pi}{4}\left(1 - e^{-2\sqrt{x}}\right). \tag{1.6}$$

This result has been given in the works [2] and [3].

2 Main Results

Theorem 2.1. Let $f(x)$ and $g(x)$ be two real functions, and the real function $c(x)$ satisfy Condition $1 - c(x-\alpha) + c(y-\alpha) \geq 0$, where $x - \alpha \geq 0$ and $y - \alpha \geq 0$. If $0 < \int_\alpha^\infty f^2(x)\, dx < +\infty$ and $0 < \int_\alpha^\infty g^2(x)\, dx < +\infty$, then

$$\left(\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)\, g(y)}{x+y-2\alpha} dxdy \right)^4 < \pi^4 \left\{ \left(\int_\alpha^\infty f^2(x)\, dx \right)^2 - \left(\int_\alpha^\infty \omega(x)\, f^2(x)\, dx \right)^2 \right\}$$

$$\times \left\{ \left(\int_\alpha^\infty g^2(x)\, dx \right)^2 - \left(\int_\alpha^\infty \omega(x)\, g^2(x)\, dx \right)^2 \right\} \tag{2.1}$$

where the weight function $\omega(x)$ is defined by

$$\omega(x) = \frac{1}{2}\left(\cos 2\sqrt{x-\alpha} - e^{-2\sqrt{x-\alpha}}\right) \qquad (2.2)$$

Proof. We firstly suppose that $f = g$. By Schwarz's inequality and then by using (1.3) and (1.4), we have

$$\left(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)\ f(y)}{x+y-2\alpha}dxdy\right)^2 = \left(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)\ f(y)}{x+y-2\alpha}\left(1-c(x-\alpha)+c(y-\alpha)\right)dxdy\right)^2$$

$$= \left(\int_\alpha^\infty\int_\alpha^\infty \left\{\frac{f(x)}{(x\ +\ y-2\alpha)^{1/2}}\left(\frac{x-\alpha}{y-\alpha}\right)^{\frac{1}{4}}\left(1-c(x-\alpha)+c(y-\alpha)\right)^{\frac{1}{2}}\right\}\right.$$

$$\left.\times\left\{\frac{f(y)}{(x\ +\ y-2\alpha)^{1/2}}\left(\frac{y-\alpha}{x-\alpha}\right)^{\frac{1}{4}}\left(1-c(x-\alpha)+c(y-\alpha)\right)^{\frac{1}{2}}\right\}dxdy\right)^2$$

$$\leq J_1 J_2$$

$$= \pi^2\left\{\left(\int_\alpha^\infty f^2(x)\,dx\right)^2 - \left(\int_\alpha^\infty k(x)\,f^2(x)\,dx\right)^2\right\}$$

where $k(x)$ is defined by (1.5).

Let's assume that $c(x) = \sin^2\sqrt{x}$. It is obvious that we have $1 - c(x) + c(y) \geq 0$.

By Lemma 1.2, it is easy to deduce that

$$k(x) = \frac{2}{\pi}\int_0^\infty \frac{c\ \left((x-\alpha)\,t^2\right)}{1+t^2}dt - c(x-\alpha) = \frac{2}{\pi}\int_0^\infty \frac{\sin^2\ \sqrt{x-\alpha}\,t}{1+t^2}dt - \sin^2\sqrt{x-\alpha}$$

$$= \frac{1}{2}\left(\cos 2\sqrt{x-\alpha} - e^{-2\sqrt{x-\alpha}}\right) = \omega(x).$$

It follows that the inequality (2.1) is valid for case $f = g$.

If $f \neq g$. By Schwarz's inequality we have

$$\left(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)\ g(y)}{x+y-2\alpha}dxdy\right)^4 = \left\{\left(\int_0^1\left(\int_\alpha^\infty t^{x-\alpha-\frac{1}{2}}f(x)\,dx\int_\alpha^\infty t^{y-\alpha-\frac{1}{2}}g(y)\,dy\right)dt\right)^2\right\}^2$$

$$\leq \left\{\int_0^1\left(\int_\alpha^\infty t^{x-\alpha-\frac{1}{2}}f(x)\,dx\right)^2 dt\right\}^2\left\{\int_0^1\left(\int_\alpha^\infty t^{y-\alpha-\frac{1}{2}}g(y)\,dy\right)^2 dt\right\}^2$$

$$= \left\{\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)\ f(y)}{x+y-2\alpha}dxdy\right\}^2\left\{\int_\alpha^\infty\int_\alpha^\infty \frac{g(x)\ g(y)}{x+y-2\alpha}dxdy\right\}^2 \qquad (2.3)$$

Using Theorem 2.1 for case $f = g$, it follows from (2.3) that the inequality (2.1) is obtained at once.

Since $f(x)\,g(x) \neq 0$, it is impossible to take equality in (2.3). Theorem is proved.

In particular, when $c(x)$ =constant, we have $1 - c(x) + c(y) = 1$, whence $k(x) = 0$. As a result, the inequality (2.1) is reduced to the inequality (1.2).

When $\alpha = 0$, we obtain a refinement of (1.1):

Corollary 2.2. Let $f(x)$ and $g(x)$ be two real functions. If $0 < \int\limits_0^\infty f^2(x)\,dx < +\infty$ and $0 < \int\limits_0^\infty g^2(x)\,dx < +\infty$, then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy\right)^4 < \pi^4 \left\{\left(\int_0^\infty f^2(x)\,dx\right)^2 - \left(\int_0^\infty \omega_0(x)\,f^2(x)\,dx\right)^2\right\} \times \left\{\left(\int_0^\infty g^2(x)\,dx\right)^2 - \left(\int_0^\infty \omega_0(x)\,g^2(x)\,dx\right)^2\right\} \tag{2.4}$$

where the weight function $\omega_0(x)$ is defined by

$$\omega_0(x) = \frac{1}{2}\left(\cos 2\sqrt{x} - e^{-2\sqrt{x}}\right) \tag{2.5}$$

In particular, when $f = g$, the following result is obtained.

Corollary 2.3. Let $f(x)$ be a real function. If $0 < \int\limits_0^\infty f^2(x)\,dx < +\infty$, then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{x+y}dxdy\right)^2 < \pi^2 \left\{\left(\int_0^\infty f^2(x)\,dx\right)^2 - \left(\int_0^\infty \omega_0(x)\,f^2(x)\,dx\right)^2\right\} \tag{2.6}$$

where the weight function $\omega_0(x)$ is defined by (2.5).

3 Applications

In this section we will give some refinements of Widder's theorem and Hardy-Littlewood's theorem with the help of Theorem 2.1 and Corollary 2.3.

Let $a_n \geq 0 (n =0, 1, 2, \ldots)$, $A(x) = \sum\limits_{n=0}^\infty a_n x^n A^*(x) = \sum\limits_{n=0}^\infty \frac{a_n x^n}{n!}$. If $A(x) \neq 0$, then

$$\int_0^1 A^2(x)dx < \pi \int_0^\infty f^2(x)dx \tag{3.1}$$

where $f(x) = e^{-x} A^*(x)$. This is Widder's theorem (see [4]).

We shall give a refinement of (3.1), below.

Theorem 3.1. With the assumptions as the above-mentioned , then

$$\left(\int_0^1 A^2(x)\,dx \right)^2 < \pi^2 \left\{ \left(\int_0^\infty f^2(x)\,dx \right)^2 - \left(\int_0^\infty \omega_0(x) f^2(x)\,dx \right)^2 \right\} \quad (3.2)$$

where $\omega_0(x)$ is defined by (2.5).

Proof. At first we have the following relation:

$$\int_0^\infty e^{-t} A^*(x)dt = \int_0^\infty e^{-t} \sum_{n=0}^\infty \frac{a_n (xt)^n}{n!} dt = \sum_{n=0}^\infty \frac{a_n x^n}{n!} \int_0^\infty t^n e^{-t} dt = \sum_{n=0}^\infty a_n x^n = A(x).$$

Let $tx = s$. Then we have

$$\int_0^1 A^2(x)dx = \int_0^1 \left(\int_0^\infty e^{-s/x} A^*(s)ds \right)^2 \frac{1}{x^2} dx = \int_1^\infty \left(\int_0^\infty e^{-sy} A^*(s)ds \right)^2 dy.$$

Let $u = y - 1$. Then

$$\int_0^1 A^2(x)dx = \int_0^\infty \left(\int_0^\infty e^{-su} f(s)ds \right)^2 du = \int_0^\infty \int_0^\infty \frac{f(s)\,f(t)}{s+t} dsdt \quad (3.3)$$

where $f(x) = e^{-x} A^*(x)$.

Using Corollary 2.3, the inequality (3.2) follows from (3.3) at once.

Let $f(x) \in L^2(0,\ 1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)\,dx, \quad n = 0,\ 1,\ 2, \cdots$$

then we have the Hardy-Littlewood's inequality (see [5])of the form

$$\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x)\,dx \quad (3.4)$$

where π is the best constant that keeps (3.4) valid. In our previous paper [6], the inequality (3.4) was extended and established the following inequality:

$$\int_0^\infty f^2(x)\,dx < \pi \int_0^1 h^2(x)\,dx \quad (3.5)$$

where $f(x) = \int_0^1 t^x h(x)\,dx,\ x \in [0,\ \ +\infty)$

The inequality (3.5) is called the Hardy-Littlewood integral inequality.

Afterwards the inequality (3.5) was refined into the following form (see [7]):

$$\int_0^\infty f^2(x)\,dx \le \pi \int_0^1 t\,h^2(t)\,dt. \tag{3.6}$$

We will further refine the inequality (3.6) here.

Theorem 3.2. Let $h(t) \in L^2(0,\ 1)$, $h(t) \ne 0$. Define a function by

$$f(x) = \int_0^1 t^{x-\alpha}\,|h(t)|\,dt(x-\alpha \ge 0)$$

If $0 < \int_\alpha^{+\infty} f^2(x)\,dx < +\infty$, then

$$\left(\int_\alpha^\infty f^2(x)\,dx\right)^2 < \pi^2\left\{\left(\int_\alpha^\infty f^2(x)\,dx\right)^2 - \left(\int_\alpha^\infty \omega(x)\,f^2(x)\,dx\right)^2\right\}\int_0^1 t\,h^2(t)\,dt. \tag{3.7}$$

where the weight function $\omega(x)$ is defined by (2.2).

Proof. Let us write $f^2(x)$ in form:

$$f^2(x) = \int_0^1 f(x)\,t^{x-\alpha}\,|h(t)|\,dt.$$

Applying in turn Schwarz's inequality and Theorem 2.1, we have

$$\left(\int_{\alpha}^{+\infty} f^2(x)\,dx\right)^2 = \left\{\int_{\alpha}^{\infty}\left(\int_0^1 f(x)\,t^{x-\alpha}\,|h(t)|\,dt\right)dx\right\}^2$$
$$= \left\{\int_0^1\left(\int_{\alpha}^{+\infty} f(x)\,t^{x-\alpha-1/2}dx\right)t^{1/2}\,|h(t)|\,dt\right\}^2$$
$$\le \int_0^1\left(\int_{\alpha}^{+\infty} f(x)\,t^{x-\alpha-1/2}dx\right)^2 dt\int_0^1 t\,h^2(t)\,dt$$
$$= \int_0^1\left(\int_{\alpha}^{+\infty} f(x)\,t^{x-\alpha-1/2}dx\right)\left(\int_{\alpha}^{+\infty} f(y)\,t^{y-\alpha-1/2}dy\right)dt\int_0^1 t\,h^2(t)\,dt$$
$$= \int_0^1\left(\int_{\alpha}^{+\infty}\int_{\alpha}^{+\infty} f(x)\,f(y)\,t^{x+y-2\alpha-1}dxdy\right)dt\int_0^1 t\,h^2(t)\,dt$$
$$= \left(\int_{\alpha}^{+\infty}\int_{\alpha}^{+\infty}\frac{f(x)f(y)}{x+y-2\alpha}dxdy\right)\int_0^1 t\,h^2(t)\,dt$$

$$\le \pi^2\left\{\left(\int_{\alpha}^{\infty} f^2(x)\,dx\right)^2 - \left(\int_{\alpha}^{\infty}\omega(x)\,f^2(x)\,dx\right)^2\right\}\int_0^1 t\,h^2(t)\,dt. \tag{3.8}$$

where the weight function $\omega(x)$ is defined by (2.2).

Since $h(t) \neq 0,\ \ f^2(x) \neq 0$. It follows that it is impossible to take equality in (3.8). We therefore complete the proof of the theorem.

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