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On Poisson Bulk Arrival Queue: $M^X/M/2/N$ with Balking, Reneging and Heterogeneous servers

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Abstract

The aim of this paper is to derive the analytical solution of the queue: $M^X/M/2/N$ for batch arrival system with balking, reneging and two heterogeneous servers. A modified queue discipline to the classical one FIFO is used with a more general condition. The steady-state probabilities and some measures of effectiveness are derived in explicit forms. Also some special cases are deduced.

Keywords: Heterogeneous servers, batch arrival, balking, reneging

1 Introduction

Abou-El-Ata and Al-Seedy [1] studied the system: $M^X/M/1$ with both balking, and reneging concepts, Cromie et al [3] discussed the queue: $M^X/M/C$ without any concepts and Al-Seedy [2] are treated the system $M/M/2/N$ with both balking and an additional server for longer queue. The present paper treats the analytical solution of the queue: $M^X/M/2/N$ with batch arrival queues considering balking, reneging, heterogeneity and different probability in choosing the servers.

2 Analyzing the model

In this work, it is assumed that the units arrive at system in batches of random size X , i.e., at each moment of arrivals there is a probability $c_j = p(x = j)$ that j units arrive simultaneously ($\sum_{j=1}^{N} cj = 1$), $c_0 = 0$ and the interarrival times of batches follow a negative exponential distribution with time independent parameter λ. Let $\lambda c_j \Delta t$, $(j = 1,....,N)$ be the first order probability that a batch of j units comes in time Δt. We assumed that we have a finite strong room such that the total number of customer in the system is no more than N and two heterogeneous servers different rates μ_1 , and μ_2 . The queue discipline considered here is modification of both Singh [5] and Krishnamoorhi [4], and it is:

i) If both servers are free, the head customer of the queue goes to the first sever with

probability π_1 or to the second sever with probability $\pi_2, \pi_1 + \pi_2 = 1$.

ii) If only one server is free, the head unit goes to directly to it.

iii) If the two servers are busy, the units in their order until any server become vacant.

Consider the balk concept with probability

$\beta =$ prob. {a unit joins the queue},

where $0 \le \beta < 1$ if $n = 2,3,....,N$ and $\beta = 1$ if $n = 0,1$.

We assume that the unit may renege according to an exponential distribution, $f(t) = \alpha e^{-\alpha t}, \quad t > 0$, with parameter α .The probability of reneging in a short period of time Δt is given by

$r_n = (n-2)\alpha \Delta t,$ for $n = 2,3,....,N$ and $r_n = 0$, for $n = 0,1,2$.

3 The steady state equation and their solution

We define the equilibrium probabilities:

$P_{0,0} =$ prob. {there is no unit in the system},

$P_{1,0} =$ prob. {there is no unit in the system},

$P_{0,1} =$ prob. {there is no unit in the system},

$P_n =$ prob. {there is no unit in the system}, $n = 2,3,....,N$

Also, $P_0 = P_{0,0}$, $P_1 = P_{1,0} + P_{0,1}$ and $P_2 = P_{1,1}$.

Using the rate out = rate in approach, the steady state difference equation can be written as follows:

$$\lambda P_0 = \mu_1 P_{1.0} + \mu_2 P_{0.1} \quad , \; n = 0 \tag{1}$$

$$\left.\begin{aligned}(\lambda+\mu_1)P_{1.0} &= \mu_2 P_{1.1} + \lambda c_1 \pi_1 P_0 \\ (\lambda+\mu_2)\; P_{0.1} &= \mu_1 P_{1.1} + \lambda c_1 \pi_2 P_0\end{aligned}\right\} \quad , n = 1 \tag{2}$$

$$(\beta\lambda + \mu_3)P_2 = (\mu_3+\alpha)P_3 + \lambda c_1 P_1 + \lambda c_2 P_0 \quad , \; n = 2 \tag{3}$$

$$\left[\beta\lambda+\mu_3+(n-2)\alpha\right]P_n = \left[\mu_3+(n-1)\alpha\right]P_{n+1} + \beta\lambda \sum_{j=1}^{n-2} c_j P_{n-j} + \lambda c_{n-1} P_1 + \lambda c_n P_0, \qquad n = 3,\; 4,\ldots,N-1 \tag{4}$$

$$\left[\mu_3+(N-2)\alpha\right]P_N = \beta\lambda\sum_{j=1}^{N-2} c_j P_{N-j} + \beta\lambda\sum_{j=2}^{N-1}\sum_{i=N-j+1}^{N-1} c_j P_i + \lambda c_N P_1 + \beta\lambda c_N \sum_{j=2}^{N-1} P_j + \lambda c_{N-1} P_1 + \lambda c_N P_0 \;, \quad n = N \tag{5}$$

where $\mu_3 = \mu_1 + \mu_2$.

From equations (1) and (2) we have:

$$P_{1,0} = \frac{\lambda\left[\lambda + \mu_2(1-c_1) + c_1\pi_1\mu_3\right]}{\mu_1\left[2\lambda+\mu_3\right]} P_0 \tag{6}$$

$$P_{0,1} = \frac{\lambda\;\left[\lambda + \mu_1(1-c_1) + c_1\pi_2\mu_3\right]}{\mu_2\left[2\lambda+\mu_3\right]} P_0 . \tag{7}$$

Therefore,

$$P_1 = \Delta P_0 , \tag{8}$$

where

$$\Delta = \frac{\lambda\left[\lambda\mu_3 + (1-c_1)(\mu_1^2+\mu_2^2) + c_1\mu_3(\pi_1\mu_2 + \pi_2\mu_1)\right]}{\mu_1\mu_2\left[2\lambda+\mu_3\right]} .$$

Summing (1) and (2) we get:

$$P_2 = \frac{\lambda}{\mu_3}\;\left[P_1 + (1-c_1)P_0\right] \tag{9}$$

We can written the steady-state equations as follows

$$P_1 = \Delta P_0 \qquad , n = 1$$

$$P_2 = \varphi\left[P_1 + (1-c_1)P_0\right] \qquad , n = 2$$

$$P_n = \left[1+\theta_{n-2}\right]P_{n-1} - \beta\varphi_{n-2}\sum_{S=2}^{n-2} c_{n-1-S} P_S - \varphi_{n-2}\left(c_{n-2}P_1 + c_{n-1}P_0\right), \qquad n = 3,\; 4,\ldots,N \tag{10}$$

where

$$\theta_n = \frac{\beta\lambda - \alpha}{\mu_3 + n\alpha},\; (n = 1,\; 2,\; \ldots\ldots, N-2), \qquad \varphi_n = \frac{\lambda}{\mu_3 + n\alpha},\; (n = \;0,\; 1,\ldots\ldots\; N-2)$$

Put $P_n = g_n P_0$ in equation (10) we deduce that:

$$g_n = \begin{cases} 1 & n=0 \\ \Delta & n=1 \\ \varphi_0\left[\Delta+(1-c_1)\right] & n=2 \\ (1+\theta_{n-2})\ g_{n-1} - \beta\varphi_{\ n-2}\sum_{s=2}^{n-2} c_{n-1-s}g_s - \varphi_{n-2}\left(c_{n-2}g_1 + c_{n-1}g_0\right) & n=3,4,\ldots,N \end{cases}$$

From the boundary condition: $\sum_{n=0}^{N} P_n = 1$, we get

$$P_0^{-1} = \left[1+\sum_{n=1}^{N} g_n\right].$$

Thus, the expected numbers of units in the system and in the queue are, respectively.

$$L = P_0 \sum_{n=1}^{N} n g_n, \qquad L_q = L - 2 + 2P_0 + P_1,$$

and the expected waiting time in the system and in the queue, respectively,

$$W = \frac{L}{\lambda'}, \quad W_q = \frac{L_q}{\lambda'},$$

where

$$\lambda' = \frac{\mu_3}{2}\ \left(L - L_q\right), \qquad \mu_3 = \mu_1 + \mu_2 .$$

4 Special cases

Some queuing systems can be obtain as special cases of this model.

i) If $\mu_1 = \mu_2$, $\pi_1 = \pi_2 = 1/2$ and $c_j = \delta_{j1}$ where δ_{j1} is the Kronecker delta function we get the homogeneous servers queue: $M/M/2/N$ with balking and reneging. Moreover if $\alpha = 0$ and $\beta = 1$, we have the queue $M/M/2/N$ without balking and reneging which studied by white et al.[8], Medhi [6] and Bunday [3].

ii) If we put $c_j = \delta_{j1}$ we obtain the heterogeneous servers queue: $M/M/2$ with balking and reneging, Moreover, if $N \to \infty$, i.e. in the infinite capacity space case, $\beta = 1$ and $\alpha = 0$ we have the queue $M/M/2$ without balking and reneging but with heterogeneous servers, which studied by Krishnanoorthi [5]. While, if $N \to \infty$, $\alpha = 0$ $\pi_1 = 1$ and $\pi_2 = 0$ we get the heterogeneous servers queue : $M/M/2$ with balking only, which discussed by singh [7].

5 Numerical Example

In the above system, letting $N=5$, i.e., the queue: $M^X/M/2/5$ the results are:

$$P_0^{-1}=1+\Delta+\varphi_0\left[\Delta+(1-c_1)\right]\left\{1+(1+\theta_1)+(1+\theta_1)(1+\theta_2)+(1+\theta_1)(1+\theta_2)(1+\theta_3)\right.$$
$$\left.-\beta\varphi_2c_1(1+\theta_3)-\beta\varphi_3c_1(1+\theta_1)-\beta\varphi_2c_1-\beta\varphi_3c_2\right\}-\varphi_1\left[c_1\Delta+c_2\right]\left\{1+(1+\theta_2)+(1+\theta_2)(1+\theta_3)-\beta\varphi_3c_1\right\}$$
$$-\varphi_2\left[c_2\Delta+c_3\right].\left\{1+(1+\theta_3)\right\}-\varphi_3\left[c_3\Delta+c_4\right],$$

$$L=P_0\left\{\Delta+\varphi_0\left[\Delta+(1-c_1)\right]\left[2+3(1+\theta_1)+4(1+\theta_1)(1+\theta_2)-4\beta\varphi_2c_1+5(1+\theta_1)(1+\theta_2)(1+\theta_3)\right.\right.$$
$$\left.-5\beta\varphi_2c_1(1+\theta_3)-5\beta\varphi_3c_1(1+\theta_1)-5\beta\varphi_3c_2\right]-\varphi_1\left[c_1\Delta+c_2\right]\left[3+4(1+\theta_2)+5(1+\theta_2)(1+\theta_3)-5\beta\varphi_3c_1\right]$$
$$\left.-\varphi_2\left[c_2\Delta+c_3\right]\left[4+5(1+\theta_3)\right]-5\varphi_3\left[c_3\Delta+c_4\right]\right\},$$

$$Lq=L+2P_0+\Delta P_0-2,$$

and the expected waiting time in the system and in the queue are, respectively,

$$W=\frac{L}{\frac{\mu_3}{2}(2-2P_0-\Delta P_0)},\qquad W_q=\frac{L_q}{\frac{\mu_3}{2}(2-2P_0-\Delta P_0)}$$

Now, we introduce the three tables for some measures of effectiveness at $\mu_1=8,\mu_2=10,\ c_1=0.23,c_2=0.22,c_3=0.21,c_4=0.18,c_5=0.16$, and $\lambda=2$ for the different values of π_1,β and α when two of them are fixed.

Table I

$\alpha = 0.05$, $\beta = 0.2$

$C_1 = 0.23$, $C_2=0.22$, $C_3=0.21$, $C_4=0.18$ and $C_5=0.16$.

Π_1	P_0	L	Lq	W	Wq
0.1	0.669043	0.690753	0.178026	0.149690	0.038579
0.2	0.668496	0.691396	0.178084	0.149659	0.038548
0.3	0.667951	0.692038	0.178141	0.149628	0.038516
0.4	0.667407	0.692679	0.178199	0.149596	0.038485
0.5	0.666863	0.693319	0.178256	0.149565	0.038454
0.6	0.666320	0.693958	0.178314	0.149534	0.038423
0.7	0.665779	0.694596	0.178371	0.149503	0.038392
0.8	0.665238	0.695232	0.178428	0.149473	0.038361
0.9	0.664698	0.695868	0.178485	0.149442	0.038331
1	0.664159	0.696503	0.178542	0.149411	0.038300

Table II

$\alpha = 0.05$, $\pi_1 = 0.2$

$C_1 = 0.23$, $C_2=0.22$, $C_3=0.21$, $C_4=0.18$ and $C_5=0.16$.

B	P_o	L	L_q	W	W_q
0.1	0.670859	0.680617	0.172560	0.148850	0.037738
0.2	0.668496	0.691396	0.178084	0.149659	0.038548
0.3	0.666114	0.702314	0.183704	0.150469	0.039358
0.4	0.663713	0.713370	0.189419	0.151280	0.040169
0.5	0.661293	0.724562	0.195230	0.152091	0.040980
0.6	0.658854	0.735890	0.201134	0.152903	0.041792
0.7	0.656398	0.747352	0.207133	0.153714	0.042603
0.8	0.653923	0.758948	0.213225	0.154525	0.043413
0.9	0.651431	0.770675	0.219410	0.155335	0.044223
1	0.648921	0.782533	0.225686	0.156144	0.045033

Table III

$\pi_1 = 0.2$, $\beta = 0.2$

$C_1 = 0.23$, $C_2=0.22$, $C_3=0.21$, $C_4=0.18$ and $C_5=0.16$.

α	P_o	L	L_q	W	W_q
0.1	0.665789	0.704708	0.185373	0.150772	0.039660
0.2	0.666486	0.701269	0.183486	0.150485	0.039374
0.3	0.667170	0.697906	0.181643	0.150205	0.039094
0.4	0.667840	0.694616	0.179843	0.149929	0.038818
0.5	0.668496	0.691396	0.178084	0.149659	0.038548
0.6	0.669141	0.688244	0.176364	0.149394	0.038282
0.7	0.669773	0.685156	0.174682	0.149133	0.038022
0.8	0.670393	0.682132	0.173037	0.148877	0.037766
0.9	0.671002	0.679168	0.171427	0.148625	0.037514
1	0.671599	0.676264	0.169852	0.148378	0.037267

6 Conclusion

In this paper, the batch arrival model: $M^X / M / 2 / N$ is studied with balking, reneging and heterogeneous servers. The recurrence relation for g_n that gives all the probabilities interms of P_0 which can be determined from the boundary condition. We discussed the example and deduced the expected number of unites in the system, in the queue, waiting time in the system and in the queue.

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