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On the Decomposition Method for System of Linear Fredholm Integral Equations of the Second Kind

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Abstract

In this paper, system of linear Freadholm integral equations of the second kind is handled by applying the decomposition method. For a system of linear equations we show that the Adomian decomposition method is equivalent to the classical successive approximations method, so called Picard's method. Finally, numerical examples are prepared to illustrate these considerations.

Keywords: Adomian decomposition method, System of linear Freadholm integral equations

1 Introduction

The topic of the Adomian decomposition method has been rapidly growing in recent years. The concept of this method was first introduced by G. Adomian in the beginning of 1980's [1, 2]. In this method the solution of a functional equations is considered as the sum of an infinite series usually converging to the solution [3].

The Adomian decomposition method for solving linear and nonlinear integral equations is known as a subject of extensive analytical and numerical studies [4, 5]. Our aim here is to compare the decomposition method with the classical successive approximations method [6] for solving system of linear Fredholm integral equations. Consider the following system of linear Fredholm integral equations:

$$F(t) = G(t) + \int_a^b K(t,s)F(s)ds, \qquad t \in [a,b] \tag{1}$$

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where

$$F(t) = (f_1(t), ..., f_n(t))^t,$$

$$G(t) = (g_1(t), ..., g_n(t))^t,$$

$$K(t,s) = [k_{i,j}(t,s)] \quad i = 1, ..., n, \quad j = 1, ..., n.$$

We suppose that system (1) has a unique solution.

2 The Decomposition Method Applied to (1)

Consider the i-th equation of (1):

$$f_i(t) = g_i(t) + \int_a^b \sum_{j=1}^{n} k_{ij}(t,s) f_j(s) ds. \tag{2}$$

From (2), we obtain *canonical form* of Adomian's equation by writing

$$f_i(t) = g_i(t) + N_i(t) \tag{3}$$

where

$$N_i(t) = \int_a^b \sum_{j=1}^{n} k_{ij}(t,s) f_j(s) ds. \tag{4}$$

To solve (3) by Adomian's method, let $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$, and $N_i(t) = \sum_{m=0}^{\infty} A_{im}$ where $A_{im},\ m = 0, 1, ...,$ are polynomials depending on $f_{10}, ..., f_{1m}, ..., f_{n0}, ..., f_{nm}$ and they are called Adomian polynomials. Hence, (3) can be rewritten as:

$$\sum_{m=0}^{\infty} f_{im}(t) = g_i(t) + \sum_{m=0}^{\infty} A_{im}(f_{10}, ..., f_{1m}, ..., f_{n0}, ..., f_{nm}). \tag{5}$$

From (4) we define:

$$\begin{cases} f_{i0}(t) = g_i(t), \\ \\ f_{i,m+1}(t) = A_{im}(f_{10}, ..., f_{1m}, ..., f_{n0}, ..., f_{nm}), \qquad i = 1, ..., n, \qquad m = 0, 1, 2, ... \end{cases} \tag{6}$$

In practice, all terms of the series $f_i(t) = \sum_{m=0}^{\infty} f_{im}(t)$ can not be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_{ik}(t) = \sum_{m=0}^{k-1} f_{im}(t), \qquad \text{with} \lim_{k\to\infty} \varphi_{ik}(t) = f_i(t). \tag{7}$$

To determine Adomian polynomials, we consider the expansions:

$$f_{i\lambda}(t) = \sum_{m=0}^{\infty} \lambda^m f_{im}(t), \tag{8}$$

$$N_{i\lambda}(f_1, ..., f_n) = \sum_{m=0}^{\infty} \lambda^m A_{im}, \tag{9}$$

where, λ is a parameter introduced for convenience. From (9) we obtain:

$$A_{im} = \frac{1}{m!}\left[\frac{d^m}{d\lambda^m} N_{i\lambda}(f_1, ..., f_n)\right]_{\lambda=0}, \tag{10}$$

and from (4), (8) and (10) we have:

$$\begin{aligned} A_{im}(f_{10}, ..., f_{1m}, ..., f_{n0}, ..., f_{nm}) &= \int_a^b \sum_{j=1}^{n} v_{ij}(s,t)\left[\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty}\lambda^l f_{jl}\right]_{\lambda=0} ds \\ &= \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_{jm} ds. \end{aligned} \tag{11}$$

So, (6) for the solution of the system of linear Fredholm integral equations will be as follow:

$$\begin{cases} f_{i0}(t) = g_i(t) & . \\ f_{i,m+1}(t) = \int_a^b \sum_{j=1}^{n} v_{ij}(s,t) f_{jm}(t) ds, \qquad i = 1, ..., n, \qquad m = 0, 1, 2, ... \end{cases} \tag{12}$$

Considering (7), we obtain:

$$\varphi_{ik}(t) = g_i(t) + \int_0^t \sum_{j=1}^{n} k_{ij}(t,s) f_{jm}(s) ds, \ i = 1, ..., n, \ m = 0, 1, 2, ... \tag{13}$$

In fact (6) is exactly the same as the well known successive approximations method for solving the system of linear Fredholm integral equations defining as:

$$f_{i,m+1}(t) = g_i(t) + \int_0^t \sum_{j=1}^{n} k_{ij}(t,s) f_{jm}(s) ds, \ i = 1, ..., n, \ m = 0, 1, 2, ... \tag{14}$$

The initial approximations for the successive approximations method is usually zero function. In other words, if the initial approximations in this method is selected $g_i(t)$, then the Adomian decomposition method and the successive approximations method are exactly the same.

3 Numerical Example

Example Consider the following system of linear Fredholm integral equations with the exact solutions $f_1(t) = t + 1$ and $f_2(t) = t^2 + 1$.

$$\begin{cases} f_1(t) = \dfrac{t}{18} + \dfrac{17}{36} + \displaystyle\int_0^1 \dfrac{s+t}{3}(f_1(s) + f_2(s))ds, \\ f_2(t) = t^2 - \dfrac{19}{12}t + 1 + \displaystyle\int_0^1 st(f_1(s) + f_2(s))ds. \end{cases}$$

To derive the solutions by using the decomposition method, we can use the following Adomian scheme:

$$\begin{cases} f_{10}(t) = \dfrac{t}{18} + \dfrac{17}{36} \simeq 0.0556t + 0.4722, \\ f_{20}(t) = t^2 - \dfrac{19}{12}t + 1 \simeq t^2 - 1.5833t + 1, \end{cases}$$

and

$$\begin{cases} f_{1,m+1}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{1m}(s) + f_{2m}(s))ds, \\ f_{2,m+1}(t) = \displaystyle\int_0^1 st(f_{1m}(s) + f_{2m}(s))ds, \qquad m = 0, 1, 2, ... \end{cases}$$

For the first iteration, we have:

$$\begin{cases} f_{11}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{10}(s) + f_{20}(s))ds = \dfrac{25}{72}t + \dfrac{103}{648} \simeq 0.3472t + 0.1590, \\ f_{21}(t) = \displaystyle\int_0^1 st(f_{10}(s) + f_{20}(s))ds = \dfrac{103}{216}t \simeq 0.4769t. \end{cases}$$

Considering (7), the approximated solutions with two terms are:

$$\begin{cases} \varphi_{12}(t) = f_{10}(t) + f_{11}(t) \simeq 0.4028t + 0.6312, \\ \varphi_{22}(t) = f_{20}(t) + f_{21}(t) \simeq t^2 - 1.1065t + 1. \end{cases}$$

Next terms are:

$$\begin{cases} f_{12}(t) = \displaystyle\int_0^1 \dfrac{(s+t)}{3}(f_{11}(s) + f_{21}(s))ds = \dfrac{185}{972}t + \dfrac{17}{144} \simeq 0.1903t + 0.1181, \\ f_{22}(t) = \displaystyle\int_0^1 st(f_{11}(s) + f_{21}(s))ds = \dfrac{17}{48}t \simeq 0.3542t. \end{cases}$$

Solutions with three terms are:

$$\begin{cases} \varphi_{13}(t) = f_{10}(t) + f_{11}(t) + f_{12}(t) \simeq 0.5931t + 0.7492, \\ \varphi_{23}(t) = f_{20}(t) + f_{21}(t) + f_{22}(t) \simeq t^2 - 0.7523t + 1. \end{cases}$$

In the same way, the components $\varphi_{1k}(t)$ and $\varphi_{2k}(t)$ can be calculated for $k = 3, 4, ...$ The solutions with eleven terms are given as:

$$\begin{cases} \varphi_{1,11}(t) = f_{10}(t) + f_{11}(t) + ... + f_{1,10}(t) \simeq 0.9813t + 0.9885, \\ \varphi_{2,11}(t) = f_{20}(t) + f_{21}(t) + ... + f_{2,10}(t) \simeq t^2 - 0.0345t + 1. \end{cases}$$

Approximated solutions for some values of t and the corresponding absolute errors are presented in Table 3.1.

t	$f_1(t)$	$\varphi_{1,11}(t)$	$e(\varphi_{1,11}(t))$	$f_2(t)$	$\varphi_{2,11}(t)$	$e(\varphi_{2,11}(t))$
0	1	0.988498	1.15×10^{-2}	1	1	0
0.1	1.1	1.086632	1.33×10^{-2}	1.01	1.006549	3.45×10^{-3}
0.2	1.2	1.184766	1.52×10^{-2}	1.04	1.033099	6.90×10^{-3}
0.3	1.3	1.282899	1.71×10^{-2}	1.09	1.079648	1.03×10^{-2}
0.4	1.4	1.381033	1.89×10^{-2}	1.16	1.146198	1.38×10^{-2}
0.5	1.5	1.479167	2.08×10^{-2}	1.25	1.232747	1.72×10^{-2}
0.6	1.6	1.577301	2.26×10^{-2}	1.36	1.339296	2.07×10^{-2}
0.7	1.7	1.675435	2.45×10^{-2}	1.49	1.465846	2.41×10^{-2}
0.8	1.8	1.773569	2.64×10^{-2}	1.64	1.612695	2.76×10^{-2}
0.9	1.9	1.871702	2.82×10^{-2}	1.81	1.778945	3.10×10^{-2}
1	2	1.969836	3.02×10^{-1}	2	1.965494	3.45×10^{-2}

Table 3.1

ii) The successive approximations method

Clearly, in this method by choosing the initial approximation of $f_1(t) = 0$ and $f_2(t) = 0$ and considering (21), in the first iteration, we will get the initial approximation of the Adomian decomposition method that is $f_{10}(t)$ and $f_{20}(t)$. In the second iteration $f_1^1(t)$ and $f_2^1(t)$ approximations for $f_1(t)$ and $f_2(t)$ are derived. And subsequently in the third iteration, we get $f_1^2(t)$ and $f_2^2(t)$ approximations for $f_1(t)$ and $f_2(t)$. Hence, if the initial approximation for $f_1(t)$ and $f_2(t)$ are respectively chosen as $g_1(t)$ and $g_2(t)$, the Adomian decomposition method and successive approximations method will be exactly the same.

4 Conclusion

This paper presents the use of the Adomian decomposition method, for the system of linear Fredholm integral equations. As it can be seen, the Adomian decomposition method for a system of linear Fredholm integral equations is equivalent to successive approximations method. Although, the Adomian decomposition method is a very powerful device for solving the functional equations, this method for a system of linear Fredholm integral equations of the second kind is not a new method.

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