Applied Mathematical Sciences, Vol. 2, 2008, no. 2, 63 - 71

Geometric Generalized Gamma Distributions and Related Processes

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Abstract

A generalization of geometric Pakes generalized Linnik laws and related processes are discussed here. This class of distributions possesses a different kind of closure under geometric compounding and consequently a different kind of p-thinning of renewal processes. Domain of geometric attraction of these laws are discussed. The convolution semi-group generator of their non-negative analogue is derived. A generalization of gamma is a closely related distribution in this discussion.

Mathematics Subject Classification: 60E05, 60E07, 60G07, 62M10

Keywords: geometric exponential laws, Pakes generalized Linnik laws, gamma laws, subordination, p-thinning, geometric compounding, convolution semi-group generator, infinite divisibility, geometric infinite divisibility

1 Introduction

Pillai (1990) had introduced geometric exponential laws with Laplace transform (LT) $\frac{1}{1+\log(1+\lambda)}$ and this was studied in some more detail in Jose and Seethalekshmi (1999). Of late various generalizations of this distribution are being studied and their roles in AR(1) models are investigated.

See *eg.* Seethalekshmi and Jose (2006) discussing geometric Pakes generalized Linnik (α, ν) law with CF $\frac{1}{1+\nu\log(1+|u|^{\alpha})}$ and the references therein. The purpose of this paper is to discuss a further generalization of the above distribution which we call geometric generalized gamma with CFs of the form $\frac{1}{1+\beta\log(1+h(u))}$, $\beta > 0$, where $e^{-h(u)}$ is a CF that is ID, and processes related to it.

Possible applications of the models considered here are in the area where geometric compounding models are being considered, *viz.* deriving non-Gaussian time series models for stock market price and in modeling finance data (Mittnick and Rachev, 1993), regenerating processes with rare events in reliability (Gertsbakh, 1984) and renewal data with missing observations (thinning) Yannaros (1987).

In section 2, we will discuss geometric generalized gamma laws and some of its divisibility properties. Restricting the discussion to the support $[0, \infty)$ we derive the convolution semi-group generator of these distributions. Using divisibility properties we discuss various processes corresponding to it in section 3.

2 Geometric generalized gamma laws

We need the following notions in our discussion. Infinitely divisible (ID) laws are well known in the literature. Klebanov, *et al.* (1984) had introduced the notion of geometrically ID (GID) laws and proved that a CF $\phi(u)$ is GID *iff* $e^{-\{\frac{1}{\phi(u)}-1\}}$ is ID. It is also known that GID laws are ID, Pillai (1990), Sandhya (1991a).

Thus corresponding to a CF $e^{-h(u)}$ that is ID we have distributions with CF $\frac{1}{1+h(u)}$ that are GID and hence ID. Hence it follows from the property of CFs which are ID that $\frac{1}{(1+h(u))^{\beta}}$, $\beta > 0$ is a CF which is ID. When $h(u) = -iu$, the above CF is that of gamma(β). Now writing this as $e^{-\beta\log(1+h(u))}$ and considering the GID law corresponding to it we get the CF $\frac{1}{1+\beta\log(1+h(u))}$, $\beta > 0$. Thus we have;

Definition 2.1 *For a CF $e^{-h(u)}$ that is ID the CF $\frac{1}{(1+h(u))^{\beta}}$, $\beta > 0$ describes a generalized gamma(β) (GG) distribution.*

Definition 2.2 *For a CF $e^{-h(u)}$ that is ID the CF $\frac{1}{1+\beta\log(1+h(u))}$, $\beta > 0$ describes a geometric generalized gamma(β) (GGG) distribution.*

Theorem 2.3 *GGG laws are GID and hence ID.*

Proof. We know that a CF $\phi(u)$ is GID *iff* $e^{-(\frac{1}{\phi(u)}-1)}$ is ID.

From our construction we have; $e^{-\beta\log(1+h(u))}$, $\beta>0$ is a CF that is ID. Now setting $\frac{1}{\phi(u)}-1=\beta\log\{1+h(u)\}$ we have;
$\dfrac{1}{1+\beta\log(1+h(u))}=\phi(u)$ is GID. Since all GID laws are ID, GGG laws are also ID.

Theorem 2.4 *A CF* $\phi(u)=\dfrac{1}{1+\log(1+h(u))^{\beta}}$, $\beta>0$ *is GGG iff* $\frac{1}{(1+h(u))^{\beta}}$, $\beta>0$ *is the CF of an ID distribution. Also GG(β) distributions are ID. This is clear from above.*

Theorem 2.5 *Every GGG(β) distribution is the limit distribution of geometric* $(\frac{1}{n})$*-sum of i.i.d GG* $(\frac{\beta}{n})$ *variables.*

Proof. Let $\phi_n(u)$ denote the CF of a geometric $(\frac{1}{n})$-sum of *i.i.d* GG $(\frac{\beta}{n})$ variables. Thus,

$$\begin{aligned}\phi_n(u)&=\frac{\frac{1}{n}(1+h(u))^{-\frac{\beta}{n}}}{1-\frac{n-1}{n}(1+h(u))^{-\frac{\beta}{n}}}\\&=\frac{1}{n(1+h(u))^{\frac{\beta}{n}}-(n-1)}\\&=\frac{1}{1+n\{(1+h(u))^{\frac{\beta}{n}}-1\}}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\phi_n(u)=\frac{1}{1+\log(1+h(u))^{\beta}}=\frac{1}{1+\beta\log(1+h(u))}$$

which proves the assertion.

Theorem 2.6 *The weak limit of n-fold convolution of GGG* $(\frac{\beta}{n})$ *laws is GG(β).*

Proof. Since

$$\lim_{n\to\infty}\{\frac{1}{1+\frac{\beta}{n}\log(1+h(u))}\}^n=e^{-\beta\log(1+h(u))}=\frac{1}{(1+h(u))^{\beta}},$$

the claim is proved.

Theorem 2.7 N_p*-sum of i.i.d GGG (β) variables is GGG* $(\frac{\beta}{p})$ *for any* $p\in(0,1)$ *where* N_p *is a geometric(p) r.v with* $E(N_p)=\frac{1}{p}$ *independent of the summands.*

Proof. The CF of geometric(p)-sum of *i.i.d* GGG (β) variables is given by;

$$\frac{p/\{1+\beta\log(1+h(u))\}}{1-(1-p)/\{1+\beta\log(1+h(u))\}} = \frac{p}{p+\beta\log(1+h(u))} = \frac{1}{1+\frac{\beta}{p}\log(1+h(u))},$$

which is the CF of a GGG ($\frac{\beta}{p}$) law.

Remark 2.8 *The invariance property of geometric(p)-sum described above is different from the invariance under geometric summation up to a scale change characterizing semi-α-Laplace distributions (Sandhya, 1991a). Here note that the shape parameter changes.*

We now discuss the domain of attraction (DA) of the GG law and the domain of geometric attraction (DGA) of the GGG law under a regularity condition. In the classical summation scheme a CF $g(u)$ belongs to the DA of the CF $f(u)$ if there exists sequences of real constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\text{as } n \to \infty, \ \{g(u/a_n)\exp(-iub_n)\}^n \to f(u) \text{ for all } u \in \mathbf{R}.$$

Setting $g_n(u) = g(u/a_n)\exp(-iub_n)$ this is equivalent to $\{g_n(u)\}^n \to f(u)$ as $n \to \infty$. When convergence is possible only if n runs through a subsequence $\{n_k\}$ of positive integers we say that the CF $g(u)$ belongs to the domain of partial attraction (DPA) of the CF $f(u)$. The notion of DGA has been developed by Sandhya (1991a) and Sandhya and Pillai (1999) and is described as follows. We have a sequence of *i.i.d r.vs* with CF $g(u)$ and set $g_p(u) = g(u/a_p)\exp(-iub_p)$, where $\{a_p > 0\}$ and $\{b_p\}$ are real constants. Then the geometric(p)-sum of $g_p(u)$ is given by $\omega_p(u) = \frac{pg_p(u)}{1-(1-p)g_p(u)}$. If $\omega_p(u) \to \omega(u)$ as $p \downarrow 0$ through $\{\frac{1}{n}\}$ then we say that the CF $g(u)$ belongs to the DGA of $\omega(u)$ and if the convergence is possible only as $p \downarrow 0$ through $\{\frac{1}{n_k}\}$ where $\{n_k\}$ is a subsequence of positive integers $\{n\}$ we say that $g(u)$ belongs to the domain of partial geometric attraction (DPGA) of $\omega(u)$. We need the following condition on the *d.f* G (Gnedenko and Korolev, 1996, p.108);

$$\int_0^\infty \omega_1^t dG(t) = \int_0^\infty \omega_2^t dG(t) \text{ implies } \omega_1 \equiv \omega_2 \text{ for any ID CFs } \omega_1 \text{ and } \omega_2. \quad (1)$$

Theorem 2.9 *Let X_1, X_2, $X_3, \ldots,$ be i.i.d r.vs with CF $g(u)$ and N_p a geometric(p) r.v with mean $\frac{1}{p}$ and independent of X_1. Let* (1) *be satisfied by the d.f of the unit exponential r.v E. Then $g(u)$ is in the DA (DPA) of GG(β) iff $g(u)$ is in the DGA (DPGA) of GGG(β).*

Proof. Since $pN_p \xrightarrow{d} E$ as $p \downarrow 0$, by Gnedenko's transfer theorem (see, Gnedenko and Korolev, 1996) it follows that if $g(u)$ is in the DA (DPA) of GG(β) law with CF $\dfrac{1}{(1+h(u))^{\beta}}$, $\beta > 0$ then $g(u)$ is in the DGA (DPGA) of GGG(β) law with CF $\frac{1}{1+\beta\log(1+h(u))}$, $\beta > 0$. Further since $N_p \xrightarrow{P} \infty$ as $p \downarrow 0$ and (1) is satisfied by the *d.f* of E, by invoking Szasz's (1972) converse to Gnedenko's transfer theorem, if $g(u)$ is in the DGA (DPGA) of GGG(β) law then $g(u)$ is in the DA (DPA) of GG(β) law.

Now we consider analogous distributions on $[0,\infty)$. LTs of the form $e^{-\psi(\lambda)}$, $\lambda > 0$ where $\psi(\lambda)$ has complete monotone derivative (CMD), $\psi(0) = 0$, Feller (1971, p.450), characterize ID laws on $[0,\infty)$. The LTs of GID laws on $[0,\infty)$ have the form $\frac{1}{1+\psi(\lambda)}$, $\lambda > 0$ where $\psi(\lambda)$ has CMD with $\psi(0) = 0$, Sandhya (1991b). Now we describe GGG laws and generalized geometric exponential (GGE) laws on $[0,\infty)$.

Definition 2.10 *A distribution with LT* $\phi(\lambda) = \dfrac{1}{1+\beta\log(1+\psi(\lambda))}$, $\beta > 0$ *and* $\psi(\lambda)$ *has CMD with* $\psi(0) = 0$ *is called a (non-negative) GGG*(ψ,β) *laws.*

Definition 2.11 *A distribution with LT* $\phi(\lambda) = \dfrac{1}{1+\log(1+\psi(\lambda))}$, *where* $\psi(\lambda)$ *has CMD with* $\psi(0) = 0$ *is called a Generalized Geometric Exponential (GGE*(ψ)*) laws.*

Remark 2.12 *When* $\psi(\lambda) = \lambda$, *we get the geometric exponential distribution. When* $\psi(\lambda) = \lambda^{\alpha}$, $0 < \alpha \leq 1$, *we get a geometric Mittag-Leffler distribution.*

The function $\psi(\lambda)$ in the LT of an ID law can be represented as $\psi(\lambda) = \int\limits_0^{\infty} \dfrac{1-e^{-\lambda x}}{x} P\{dx\}$, where P is a measure such that $\int\limits_1^{\infty} \dfrac{P\{dx\}}{x} < \infty$, (Feller, 1971, p.450). This measure is also known as the generator of the convolution semi-group corresponding to the ID law, (see Feller, 1971, example (a) on p.457, 458). Next we derive the convolution semi-group generator of GGG(ψ,β).

Theorem 2.13 *Let* X *be a GG*(ψ,β) *and* Y *be GGG*(ψ,β). *Let* P_1 *be the convolution semi-group generator of* X *and* P_0 *that of* Y. *Let* F *be the d.f of* X *and* G *that of* Y. *Then* $P_0 = G * P_1$

Proof. Setting $(1+\psi(\lambda))^{-\beta} = \exp\{-\beta\log(1+\psi(\lambda))\} = \exp\{-\psi_1(\lambda)\}$, we have;

$$\psi_1(\lambda) = \int_0^{\infty} \frac{1-e^{-\lambda x}}{x} P_1\{dx\} \quad \text{and hence } \psi_1'(\lambda) = \int_0^{\infty} e^{-\lambda x} P_1\{dx\}.$$

Now setting $\{1+\beta\log(1+\psi(\lambda))\}^{-1} = \exp\{-\log\{1+\beta\log(1+\psi(\lambda))\}\} = \exp(-\psi_0(\lambda))$
we have; $\int_0^\infty e^{-\lambda x}P_0\{dx\} = \psi_0'(\lambda) = \dfrac{1}{1+\beta\log(1+\psi(\lambda))}\psi_1'(\lambda)$.
This implies that $P_0 = G * P_1$.

When $\beta = 1$, we have; $\psi_0'(\lambda) = \dfrac{1}{1+\log(1+\psi(\lambda))}\dfrac{1}{\{1+\psi(\lambda)\}}\psi'(\lambda)$ and so;

Corollary 2.14 *Let X be a $GG(\psi, 1)$ and Y be $GGE(\psi)$. Let P be the convolution semi-group generator of the ID distribution with LT $e^{-\psi(\lambda)}$, P_1 that of X and P_0 that of Y. Let F be the d.f of X and G that of Y. Then $P_0 = G * F * P$.*

This extends theorem 2.1 in Pillai and Sandhya (2001).

Remark 2.15 *It is quite interesting to note that for a non-negative function $\psi(\lambda)$ having CMD with $\psi(0) = 0$, $\log(1+\psi(\lambda))$ also has CMD. Further starting from a $\psi(\lambda)$ having CMD we can recursively take $\log(1+\psi(\lambda))$ to have functions with CMD. Eg. If $\psi(\lambda)$ has CMD then $\log\{1+\log[1+\log(1+\psi(\lambda))]\}$ also has CMD. CFs can also be derived recursively as above. Thus various LTs of probability distributions can be derived.*

3 Processes related to GGG and GG laws

3.1 Subordination of Levy processes

From Feller (1971, p.573) we have: Let $\{Y(t), t \geq 0\}$ be a Levy process with CF $\exp\{-th(u)\}$ and $\{T(t), t \geq 0\}$ a positive Levy process independent of $\{Y(t)\}$ with LT ϕ^t . Then the process $\{X(t), t \geq 0\}$ is said to be subordinated to $\{Y(t)\}$ by $\{T(t)\}$, the directing process, if $\{X(t)\} \stackrel{d}{=} \{Y(T(t))\}$ and the CF of $\{X(t)\}$ is given by $f(u) = \{\phi(h(u))\}^t$. Here we are randomizing the time parameter t of $\{Y(t)\}$ by $\{T(t)\}$. Since GGG laws and GG laws are ID they can describe Levy processes and we have the following two results.

Theorem 3.1 *A GGG (β) process is subordinated to a GG(β) process by the unit exponential process.*

Proof. If $\{Y(t)\}$ is a GG(β) process and $\{T(t)\}$ is unit exponential with *d.f* G then:

$$\frac{1}{1+\beta\log(1+h(u))} = \int_0^\infty e^{-t\beta\log(1+h(u))}dG(t),\ \text{which proves the assertion.}$$

Theorem 3.2 *Let $X(t)$ be a Levy process with $e^{-h(u)}$ as the CF of $X(1)$. Then the GGG (β) process is subordinated to $X(t)$ by the GG(β) process.*

Proof. We know that the LT of a GG(β) law is $\dfrac{1}{1+\log(1+\lambda)^{\beta}}$. If G denotes its *d.f* then we have;

$$\frac{1}{1+\beta\log(1+h(u))} = \int_0^\infty e^{-th(u)}dG(t), \quad \text{proving the assertion.}$$

This generalizes theorem 2.1 in Seethalekshmi and Jose (2001).

3.2 p-thinning of renewal processes

Let the *i.i.d* sequence of non-negative *r.v*s $\{X_i\}$ describe the inter-arrival times of a renewal process. Suppose that every renewal point of this process is retained with a constant probability p and deleted with probability $(1-p)$ independent of all other points and the process itself. The resulting process is called the p-thinned process of $\{X_i\}$. See, Renyi (1956), Yannaros (1987) or Sandhya (1991b). If an *i.i.d* sequence of non-negative *r.v*s $\{Y_i\}$ describe the inter-arrival times of the p-thinned process, then the LTs of X_i and Y_i are related by

$$\phi_Y(\lambda) = \frac{p\phi_X(\lambda)}{1-(1-p)\phi_X(\lambda)} \text{ for some } 0<p<1.$$

Now invoking theorem 2.7 we get;

Theorem 3.3 *A GGG (ψ,β) renewal process is invariant under p-thinning for any $p\in(0,1)$. Here the p-thinned process is described by the GGG (ψ, $\frac{\beta}{p}$) variable.*

3.3 An Auto-regressive model

Now consider the first order autoregressive (AR(1)) model (2.2) of Lawrance and Lewis (1981). Here a sequence of *r.v*s $\{X_n, n>0 \text{ integer}\}$ defines the AR(1) scheme if for some $0<p<1$ there exists an innovation sequence $\{\epsilon_n\}$ of *i.i.d* *r.v*s such that;

$$X_n = \begin{cases} \epsilon_n, & \text{with probability } p \\ X_{n-1}+\epsilon_n, & \text{with probability (1-p).} \end{cases} \tag{2}$$

Assuming stationarity in terms of CFs this is equivalent to;

$$\phi_X(u) = p\phi_\epsilon(u) + (1-p)\phi_X(u)\phi_\epsilon(u).$$

That is;

$$\phi_X(u) = \frac{p\phi_\epsilon(u)}{1-(1-p)\phi_\epsilon(u)}.$$

Hence $\{X_n\}$ is a geometric sum of the innovation sequence $\{\epsilon_n\}$. Once again invoking theorem 2.7 we have proved;

Theorem 3.4 *In the AR(1) structure (2) the sequence $\{X_n\}$ and the innovation sequence $\{\epsilon_n\}$ are related as follows for any $p \in (0,1)$. $\{X_n\}$ is GGG(ψ, β) iff $\{\epsilon_n\}$ is GGG ($\psi, \frac{\beta}{p}$).*

Equivalently,

Theorem 3.5 *A necessary and sufficient condition for an AR(1) process $\{X_n\}$ with the structure in (2) is stationary for any $p \in (0,1)$ with GGG (ψ, β) distributed marginals is that the innovation's are distributed as GGG($\psi, \frac{\beta}{p}$).*

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Received: June 4, 2007