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New Alternative to Formulate Utility Functions in Consumer Theory

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Abstract

In this paper, we propose a mathematical approach leading systematically to a key result of the consumer theory: the representation of utility function.

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1 Introduction

In the microeconomic area, the consumer theory presents several open questions related to the utility function. This function is introduced to measure the preferences of consumer in the sense that one good $y \geq 0$ is preferred to an other good x if and only if $u(x) \leq u(y)$. The consumer problem is formulated as follows:

$$\max \{u(x) \,:\, x \in \mathbb{R}^n_+, \;\; \langle p, x \rangle \leq b\} \qquad (U)$$

where x is the commodity bundle, $p > 0$ is the vector of unitary prices and $b > 0$ is the budget of the consumer. We are concerned with the form of utility function u. Our study uses a mathematical approach witch permit to find naturally the form and fundamentals properties of utility function.

In all paper, the symbol $\diamond$ means the end of proof.

2 Projection Theorem

Definition 1 *A subset C of $\mathbb{R}^n$ is convex if:*

$$(1-\lambda)x + \lambda y \in C, \forall\, x, y \in C \;\; \text{and} \; \forall\, \lambda \in [0, 1]$$

it means that the segment $[x, y] = \{(1-\lambda)x + \lambda y \ / \ 0 \leq \lambda \leq 1\}$ *is entierly included in* C.

There are several manners to present the projection theorem. In our context, it is convenient to formulate this theorem as a consequence of the following classical result:

Theorem 1 *Consider the convex optimization prolem*

$$(pc) \qquad \begin{cases} \min f(x) \\ x \in C \end{cases}$$

where f *is a convex differentiable function from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *and* C *is a closed convex non empty subset in* $\mathbb{R}^n$. *Then,* $\bar{y} \in C$ *is an optimal solution of* (pc) *if and only if*

$$\langle \nabla f(\bar{y}), x - \bar{y} \rangle \geq 0 \ , \ \forall x \in C$$

where ∇f *is the gradient of* f.

Theorem 2 *Let* C *be a closed convex non empty subset of* $\mathbb{R}^n$ *and let* y *be any element of* $\mathbb{R}^n$ *not lying in* C, *then there exist a unique element* $\bar{y} \in C$ *such that:*

$$\langle y - \bar{y}, x - \bar{y} \rangle \leq 0 \ , \ \forall x \in C$$

$\bar{y}$ *is called the orthogonal projection of* y *on* C, *we write* $\bar{y} = P_C(y)$.

Note that this result is a direct consequence of theorem1 taking
$f(x) = \frac{1}{2} \|x - y\|^2$.

3 Normal Cone

Definition 2 *Let* a *be a point of* $C \subset \mathbb{R}^n$, *the normal cone of* C *at* a *is the set:*

$$\mathbb{N}_C(a) = \{d \in \mathbb{R}^n : \langle d, x - a \rangle \leq 0, \forall x \in C\}$$

Lemma 1 *Given a convex closed non empty subset* C *of* $\mathbb{R}^n$ *and an element* a *of* C. *We have the characterization*
$d \in \mathbb{N}_C(a)$ *if and only if* $a = P_C(a + d)$.

Proof : According to projection theorem we have

$$\begin{aligned} a = P_C(a+d) &\Longleftrightarrow \langle a + d - a, x - a \rangle \leq 0 \ \forall x \in C \\ &\Longleftrightarrow \langle d, x - a \rangle \leq 0 \ \forall x \in C. \\ &\Longleftrightarrow d \in \mathbb{N}_C(a) \ \diamond \end{aligned}$$

Proposition 1 *Let $C = \left\{x \in \mathbb{R}^n : p^T x \leq b\right\}$ be a half space and $a \in C$ then:*

$$\mathbb{N}_C(a) = \begin{cases} \{0\} & if \ p^T a < b \\ \{\lambda b : \lambda \geq 0\} & if \ p^T a = b \end{cases}$$

Proof : We have from lemma1:

$d \in \mathbb{N}_C(a) \iff a = p_C(a+d)$ hence: a is the unique solution of the convex quadratic program:

$$(q) \qquad \begin{cases} \min f(x) = \frac{1}{2}\left\|x - a - d\right\|^2 \\ p^T x \leq b \iff g(x) = p^T x - b \leq 0 \end{cases}$$

for witch the necessary and sufficient optimality conditions of Karush-Kuhn-Tucker are:

$\exists \lambda \geq 0 \quad (\lambda \in \mathbb{R})$ such that: $\begin{cases} \nabla f(a) + \lambda \nabla g(a) = 0 \\ \lambda g(a) = 0 \end{cases}$

where $\begin{cases} \nabla f(x) = x - a - d \\ \nabla g(a) = p \end{cases}$

by substitution we obtain

$$\begin{cases} -d + \lambda p = 0 \\ \lambda(p^T a - b) = 0 \end{cases}$$

$$\iff \quad \begin{cases} d = \lambda P, \quad \lambda \geq 0 \\ \lambda(p^T a - b) = 0 \end{cases}$$

If $(p^T a - b) < 0$ then $\lambda = 0$ and so $d = 0$

Else $(p^T a - b) = 0$ then $\lambda \geq 0$ and $d = \lambda P$ $\diamond$

4 Convex Function

Definition 3 *Let f be a fonction from $\mathbb{R}^n$ to $\mathbb{R}$, then f is convex if and only if: $\forall\, x, y \in \mathbb{R}^n, \forall\, \lambda \in [0,1]$*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Definition 4 *Let $x_0 \in \mathbb{R}^n$ such that $f(x_0)$ is finite. Then the set*

$$\partial f(x_0) = \{g \in \mathbb{R}^n / f(x) \geq f(x_0) + \langle g, x - x_0\rangle, \forall x \in \mathbb{R}^n\}$$

is called ***subdifferential*** *of f at x_0. Any element $g_0 \in \partial f(x_0)$ is called* ***subgradient*** *of f at x_0.*

Property 1 Let $x_0 \in C \subset \mathbb{R}^n$ then

$\partial\psi_C(x_0) = \mathbb{N}_C(x_0)$ where ψ_C is the indicator function of the set C defined by:

$$\psi_C(x) = \begin{cases} 0 & if\ x \in C \\ +\infty & if\ x \notin C \end{cases}$$

Property 2 Let f_1 and f_2 be two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ with finite values at x_0 then the function $f = f_1 + f_2$ is convex and we have

$$\partial f(x_0) = \partial f_1(x_0) + \partial f_2(x_0)$$

5 Concave Function

Definition 5 *Let u be a function defined from $\mathbb{R}^n$ to $\mathbb{R}$, then u is concave if and only if:* $\forall x, y \in \mathbb{R}^n, \forall \lambda \in [0,1]$

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$$

Definition 6 *Let u be a concave function finite at $\bar{x} \in \mathbb{R}^n$. The set*

$$\bar{\partial} u(\bar{x}) = \{g \in \mathbb{R}^n / \, u(x) \leq u(\bar{x}) + \langle g, x - \bar{x}\rangle, \forall x \in \mathbb{R}^n\}$$

is called ***superdifferential*** *of u at $\bar{x}$.*

An element $\bar{g} \in \bar{\partial} u(\bar{x})$ is called ***supergradient*** *of u at $\bar{x}$.*

Note that $-u$ is convex and we have trivially the important relation:

Property 3 $\bar{\partial} u(\bar{x}) = -\, \partial(-u)(\bar{x}) \quad \forall\, \bar{x} \in \mathbb{R}^n$

The following representation of a concave function is fundamental for the sequel.

Proposition 2 *Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a concave function, then we have:*

$$u(x) = \min_{z \in \mathbb{R}^n} \{u(z) + \langle g(z), x - z\rangle\}$$

where $g(z)$ is an arbitrary element of $\bar{\partial} u(z)$. The minimum can be also taken on any open subset of $\mathbb{R}^n$ containing x.

Proof : $g(z) \in \bar{\partial} u(z) \Longrightarrow u(x) \leq u(z) + \langle g(z), x - z \rangle$ this is true $\forall\, z \in \mathbb{R}^n$ so

$$u(x) \leq \inf_{z \in \mathbb{R}^n} \{u(z) + \langle g(z), x - z \rangle\}$$

The relation

$$\inf_{z \in \mathbb{R}^n} \{u(z) + \langle g(z), x - z \rangle\} \leq u(x)$$

is trivial. And therefore the equality

$$u(x) = \inf_{z \in \mathbb{R}^n} \{u(z) + \langle g(z), x - z \rangle\}$$

holds. The infimum is attained at $z = x$, hence

$$u(x) = \min_{z \in \mathbb{R}^n} \{u(z) + \langle g(z), x - z \rangle\} \ \diamond$$

6 Maximization of a concave function on a half-space

Consider the optimization problem

$$(M) \begin{cases} \max u(x) \\ x \in C \end{cases}$$

where $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous concave function, and C a half-space of the form

$$C = \left\{ x \in \mathbb{R}^n : p^T x \leq b \right\}$$

The problem (M) is equivalent to

$$(m) \begin{cases} \min - u(x) \\ x \in C \end{cases}$$

or also

$$\min \{- u(x) + \psi_C(x) : x \in \mathbb{R}^n\}$$

Because this last is convex and without contraints, a necessary and sufficient condition for $\bar{x} \in C$ to be an optimal solution is that

$$0 \in \partial(- u + \psi)(\bar{x}) \qquad (CNS)$$

According to properties 1, 2 and 3, (CNS) is written as $0 \in \left[\bar{\partial} u(\bar{x}) + \mathbb{N}_C(\bar{x})\right]$
In other words, there exist $g \in \bar{\partial}(\bar{x})$ and $\lambda \geq 0$ such that $0 = - g + \lambda p$ so $g = \lambda p$

7 Application

From proposition 2 and optimality condition (CNS) ,we have
$\forall\, x, \bar{x} \in \mathbb{R}^n,\ \exists\, \lambda(\bar{x}) \geq 0$ such that

$$u(x) = \min_{\bar{x} \in \mathbb{R}^n} \{u(\bar{x}) + \lambda(\bar{x})\langle p, x - \bar{x}\rangle\}$$

In practice, we have only a finite number of points (observations) $(x^i, p^i)_{i \in I},\ I = \{1, 2, \cdots, m\}$, in other words, we know only a finite number of values of u. From this, u will be represented as follows:

$$u(x) = \min_{i \in I} \left\{u(x^i) + \lambda_i \langle p^i, x - x^i\rangle\right\}$$

We recognize here the classical formulation of utility function

$$u(x) = \min_{i \in I} \left\{\varphi_i + \lambda_i \langle p^i, x - x^i\rangle\right\}$$

where the quantities $\varphi_i \in \mathbb{R}$ and $\lambda_i \geq 0$ satisfy Afriat linear inequalities system

$$\varphi_j \leq \varphi_i + \lambda_i a_{ij}, \forall i, j \in I\ \ ,\ a_{ij} = \langle p^i, x^j - x^i\rangle$$

and can be determined by solving the following linear program [5]:

$$\left\{\begin{array}{c} \min_{\lambda,\varphi} \sum_{i=1}^{m} w_i \lambda_i \\ a_{ij}\, \lambda_i + \varphi_i - \varphi_j \geq 0,\ \text{for all}\ i, j \in I\ ,\ i \neq j \\ \lambda_i \geq 1,\ \text{for all}\ i \in I \end{array}\right. \qquad (b_w)$$

with $w \in \mathbb{R}^m, w \neq 0$.
Different w lead to different functions u.
Note that we have $\forall\, j \in I\,,\, \varphi_j = u(x^j)$, it is a fondamental property that function u must verify to generate the observations.

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