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New Infinite Product Representations of Some Elementary Functions

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Abstract

The derivation of Green's functions is revisited in a trivial case of standard boundary value problems for the two-dimensional Laplace equation. Regions of a regular shape are considered, with Dirichlet and/or Neumann boundary conditions imposed. Classical closed analytic form of Green's functions are reviewed and the method of images is used for obtaining their alternative representations in terms of infinite products. The latter are obviously less attractive compared to the closed form of Green's functions. But the point, however, is that a surprising aspect was discovered when the two forms are compared. This brings some new 'summation' formulae for infinite products leading, in turn, to unlooked-for results in the approximation of elementary functions.

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1 Introduction

The method of images [2,4,6] represents one of the classical approaches to the construction of Green's functions for Laplace equation. Procedure of this method is unpretentious. The idea behind is to find, for any location of a unit source inside the region, a location and intensity of point sources outside the region in such a way that homogeneous boundary conditions imposed on the regions's boundary are satisfied. The number of problems, for which the method of images turned out productive, is limited to a few. It works only for several particular problems posed on regions of standard configuration, with either Dirichlet of Neumann boundary conditions imposed.

It is, for instance, evident that the classical [4] Green's function

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\sqrt{\frac{(x-\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}} \tag{1}$$

of the Dirichlet problem for the half-plane $y>0$ represents the sum of two components

$$-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}$$

and

$$\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y+\eta)^2}$$

the first of which is the fundamental solution of the Laplace equation in two dimensions and represents the response at an arbitrary field point (x,y) to a unit source located at (ξ,η), while the second component is the response to a unit sink located at the point $(\xi,-\eta)$. The latter is the image of (ξ,η) about the boundary line of the half-plane.

As another example of successful application of the method of images, we consider the function

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\left(\sqrt{(x-\xi)^2+(y-\eta)^2}\right.$$
$$\left.\times\sqrt{(x-\xi)^2+(y+\eta)^2}\right) \tag{2}$$

which is the sum of responses to two unit sources located at (ξ,η) and $(\xi,-\eta)$. The above represents Green's function of the Neumann problem for the half-plane $y>0$.

Upon combining two unit sources located at (ξ,η) and $(-\xi,-\eta)$ with two unit sinks placed at $(-\xi,\eta)$ and $(\xi,-\eta)$, one arrives at the Green's function

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\left(\sqrt{\frac{[(x-\xi)^2+(y+\eta)^2]}{[(x-\xi)^2+(y-\eta)^2]}}\right.$$
$$\left.\times\sqrt{\frac{[(x+\xi)^2+(y-\eta)^2]}{[(x+\xi)^2+(y+\eta)^2]}}\right) \tag{3}$$

of the Dirichlet problem posed on the quarter-plane $\{x>0,\ y>0\}$.

Relocating the sources and the sinks in (3), one arrives at the Green's function

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\left(\sqrt{\frac{[(x-\xi)^2+(y+\eta)^2]}{[(x-\xi)^2+(y-\eta)^2]}}\right.$$

$$\times \sqrt{\frac{[(x+\xi)^2+(y+\eta)^2]}{[(x+\xi)^2+(y-\eta)^2]}}\Bigg) \tag{4}$$

for a mixed boundary value problem on the quarter-plane $\{x > 0,\ y > 0\}$, where Dirichlet condition is assumed on the fragment $y = 0$ of the boundary, while Neumann condition is imposed on the fragment $x = 0$.

The Green's function [5]

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\ln\frac{1}{R}\sqrt{\frac{R^4-2R^2r\varrho\cos(\varphi-\psi)+r^2\varrho^2}{r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2}} \tag{5}$$

of the Dirichlet problem for a disc of radius R centered at the origin, can be obtained, by the method of images, as the sum of the response

$$-\frac{1}{2\pi}\ln\sqrt{r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2}$$

to a unit source placed at an arbitrary point (ϱ,ψ) inside the disc and to a sink placed outside the disc, response to which is given as

$$\frac{1}{2\pi}\ln\frac{1}{R}\sqrt{R^4-2R^2r\varrho\cos(\varphi-\psi)+r^2\varrho^2}$$

The expressions for Green's functions listed above represent just a few ones whose compact closed form is obtainable by the method of images. The present study discloses a surprising outcome from the procedure of this method when it is used for the derivation of alternative forms for some classical Green's functions. This leads to a different research area, making it possible to obtain new infinite product representations for some elementary functions.

2 Dirichlet problem for an infinite strip

We begin, in this section, a revision of some classical Green's functions for Laplace equation, which have traditionally been obtained by means of other methods, and obtain then those Green's functions by the method of images. This allows us to derive new 'summation' formulae for some infinite functional products.

For the first example, consider Dirichlet problem for Laplace equation stated on the infinite strip $\Omega = \{-\infty < x < \infty,\ 0 < y < b\}$. Green's function

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\sqrt{\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}},\quad \omega=\frac{\pi}{b} \tag{6}$$

for this problem can be found in standard texts on partial differential equations. It was obtained in [3], for example, by a modified version of the method of eigenfunction expansion [6]. That version brings computer-friendly forms of Green's functions due to either complete or partial summation of their series representations.

Another alternative expression for the Green's function shown in (6) can be obtained by the method of images. In doing so, we place a unit source S_0^+ at an arbitrary point $(\xi,\eta)\in\Omega$. The response to S_0^+ is the fundamental solution

$$G_0^+(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}$$

To compensate traces of S_0^+ on the boundary fragments $y=0$ and $y=b$, we place two unit sinks $S_{1,0}^-$ and $S_{1,b}^-$ at the points $(\xi,-\eta)$ and $(\xi,2b-\eta)$, which are the images of (ξ,η) about the lines $y=0$ and $y=b$, respectively. The responses to these sinks are

$$G_{1,0}^-(x,y;\xi,-\eta) = \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y+\eta)^2}$$

and

$$G_{1,b}^-(x,y;\xi,2b-\eta) = \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(2b-\eta))^2}$$

Traces of the sinks $S_{1,0}^-$ and $S_{1,b}^-$ on the boundary lines $y=0$ and $y=b$ can, in turn, be compensated with unit sources $S_{2,0}^+$ and $S_{2,b}^+$ which are located at $(\xi,-2b+\eta)$ and $(\xi,2b+\eta)$. The responses to these are given as

$$G_{2,0}^+(x,y;\xi,-2b+\eta) = -\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(-2b+\eta))^2}$$

and

$$G_{2,b}^+(x,y;\xi,2b-\eta) = -\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(2b+\eta))^2}$$

Traces of the sources $S_{2,0}^+$ and $S_{2,b}^+$ can then be compensated with unit sinks $S_{3,0}^-$ and $S_{3,b}^-$ located at $(\xi,-2b-\eta)$ and $(\xi,4b-\eta)$.

Following the described procedure of properly placing compensatory unit sources that alternate with unit sinks, the Green's function $G=G(x,y;\xi,\eta)$ that we are looking for is obtained in a form of the infinite series

$$G = G_0^+ + \sum_{i=1}^{\infty}\left(G_{2i-1,0}^- + G_{2i-1,b}^-\right) + \sum_{i=1}^{\infty}\left(G_{2i,0}^+ + G_{2i,b}^+\right)$$

Since the terms of this series represent logarithmic functions, its sum can be written as a single logarithm of an infinite product which is ultimately found in the form

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}} \tag{7}$$

Thus, (7) delivers a new representation of the Green's function for the Dirichlet problem on the infinite strip, which is alternative to that in (6). Since the radicands in (6) and (7) are non-negative quantities, one immediately obtains

$$\prod_{n=-\infty}^{\infty} \frac{(x-\xi)^2 + [y + (\eta - 2nb)]^2}{(x-\xi)^2 + [y - (\eta - 2nb)]^2}$$
$$= \frac{1 - 2e^{\omega(x-\xi)} \cos \omega (y+\eta) + e^{2\omega(x-\xi)}}{1 - 2e^{\omega(x-\xi)} \cos \omega (y-\eta) + e^{2\omega(x-\xi)}}$$

This can be interpreted as a 'summation' formula for the infinite product. Assuming $b = \pi$, in the above relation, and introducing the parameters $\beta = x - \xi$, $u = y + \eta$ and $v = y - \eta$, we obtain the identity

$$\prod_{n=-\infty}^{\infty} \frac{\beta^2 + (u - 2n\pi)^2}{\beta^2 + (v + 2n\pi)^2} = \frac{1 - 2e^{\beta} \cos u + e^{2\beta}}{1 - 2e^{\beta} \cos v + e^{2\beta}} \tag{8}$$

Based on the fact that both the observation point (x, y) and the source point (ξ, η) belong to Ω, the identity in (8) is supposed to be valid (at least, formally) for

$$-\infty < \beta < \infty, \qquad 0 < u < 2\pi, \qquad 0 \leq v < \pi \tag{9}$$

given that the parameters β and v are not equal zero at the same time. But it is important to note that if the product in (8) appears to be uniformly convergent for a wider range of the variables u and v, then the constraints on these variables in (9) ought to be accordingly revised.

The identity in (8) reduces to

$$\prod_{n=-\infty}^{\infty} \frac{(u - 2n\pi)^2}{(v + 2n\pi)^2} = \frac{1 - \cos u}{1 - \cos v} = \sin^2 \frac{u}{2} \csc^2 \frac{v}{2}$$

if we assume a zero value for the parameter β (with $v \neq 0$).

It is evident that the above identity holds if

$$\prod_{n=-\infty}^{\infty} \frac{u - 2n\pi}{v + 2n\pi} = \sin \frac{u}{2} \csc \frac{v}{2} \tag{10}$$

representing an infinite product expansion of the function

$$F(u, v) = \sin \frac{u}{2} \csc \frac{v}{2}$$

To take a closer look at the convergence of the infinite product in (10), we isolate the term $n = 0$, which is u/v, and group the terms $n = k$ and $n = -k$. This yields

$$\prod_{n=-\infty}^{\infty} \frac{u - 2n\pi}{v + 2n\pi} = \frac{u}{v} \prod_{k=1}^{\infty} \frac{(u - 2k\pi)(u + 2k\pi)}{(v + 2k\pi)(v - 2k\pi)}$$

$$= \frac{u}{v}\prod_{k=1}^{\infty}\frac{u^2-4k^2\pi^2}{v^2-4k^2\pi^2}$$

$$= \frac{u}{v}\prod_{k=1}^{\infty}\frac{u^2-v^2+v^2-4k^2\pi^2}{v^2-4k^2\pi^2}$$

$$= \frac{u}{v}\prod_{k=1}^{\infty}\left(1+\frac{u^2-v^2}{v^2-4k^2\pi^2}\right)$$

This form of the product implies [1] that it uniformly converges as soon as the series

$$\sum_{k=1}^{\infty}\frac{u^2-v^2}{v^2-4k^2\pi^2}$$

does so. But the above series represents a generalized harmonic series with the convergence rate of the order of $1/k^2$. It uniformly converges [5] for any finite value of u and v. Based on that it is possible to conclude that the constraints put on the parameters u and v in (9) can be revised. This, in turn, implies that the product in (10) uniformly converges to a value of the function $F(u,v)$ at any point (u,v) in its domain.

Two infinite product representations for single-variable trigonometric functions can be obtained from that in (10). Indeed, if we assume $v=\pi$ and make a substitution $u/2=t$, then the relation in (10) transforms in the expansion

$$\sin t = \prod_{n=-\infty}^{\infty}\frac{2(t-n\pi)}{(2n+1)\pi} \tag{11}$$

of the sine function in an infinite product, the uniform convergence of which evidently follows from the analysis that we just completed for the infinite product in (10).

The expansion in (11) can be transformed, similarly to that in (10), by isolating the $n=0$ term, which is $2t/\pi$, and coupling the $n=k$ and $n=-k$ terms. This yields

$$\sin t = \frac{2t}{\pi}\prod_{k=1}^{\infty}\frac{4(t^2-k^2\pi^2)}{(1-4k^2)\pi^2}$$

which, after a trivial algebra, reads

$$\sin t = \frac{2t}{\pi}\prod_{k=1}^{\infty}\left[1+\frac{4t^2-\pi^2}{(1-4k^2)\pi^2}\right] \tag{12}$$

The expansion in (12) represents an alternative to the classical [1] infinite product form

$$\sin t = t\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2\pi^2}\right) \tag{13}$$

of the sine function.

It is evident that the products in (12) and (13) converge at the same rate. This assertion follows from the appearance of their general terms. Indeed, both of them converge to the unity value at the rate of $1/k^2$. It appears from our observation, however, that the actual convergence of the product in (12) is somewhat higher of that in (13). This observation does not, of course, conflict with the *apriori* estimate, but rather gives a comparison of the practical convergence. The latter is illustrated with the data in Table 1, where to give a clear view of the convergence rate of both representations, we display the relative error of their K-th partial products

$$\prod\nolimits_K^{(12)} = \frac{2t}{\pi} \prod_{k=1}^{K} \left[1 + \frac{4t^2 - \pi^2}{(1-4k^2)\pi^2} \right]$$

and

$$\prod\nolimits_K^{(13)} = t \prod_{k=1}^{K} \left(1 - \frac{t^2}{k^2\pi^2} \right)$$

computed for a few values of t.

The infinite product representation

$$\cos t = \sin\left(\frac{\pi}{2} - t\right) = \frac{\pi - 2t}{\pi} \prod_{k=1}^{\infty} \left[1 + \frac{4t(t-\pi)}{(1-4k^2)\pi^2} \right] \tag{14}$$

for the cosine function, as directly obtained from (12), is an alternative to the classical [1] form

$$\cos t = \prod_{k=1}^{\infty} \left(1 - \frac{4t^2}{(2k-1)^2\pi^2} \right)$$

Going back to the relation in (10), letting $u = \pi$ and making the substitution $v/2 = t$ yields the following representation

$$\csc t = \prod_{n=-\infty}^{\infty} \frac{(1-2n)\pi}{2(t+n\pi)} = \prod_{n=-\infty}^{\infty} \left[-1 + \frac{2t+\pi}{2(t+n\pi)} \right] \tag{15}$$

for the cosecant function. The infinite product in (15) uniformly converges to values of the cosecant at any point in its domain.

From the representations in (11) and (15) it follows

$$\prod_{n=-\infty}^{\infty} \frac{2(t-n\pi)}{(2n+1)\pi} \equiv \prod_{n=-\infty}^{\infty} \frac{2(t+n\pi)}{(1-2n)\pi}$$

The equivalence of these products is evident because each of them is indifferent to the replacement of n with $-n$.

Upon revisiting the relation in (8) and assuming $u = 0$ and $v = \pi$ in it, one obtains

$$\prod_{n=-\infty}^{\infty} \frac{\beta^2 + 4n^2\pi^2}{\beta^2 + (1+2n)^2\pi^2} = \frac{(1-e^{\beta})^2}{(1+e^{\beta})^2}$$

This can be rewritten in terms of a hyperbolic function as

$$\tanh^2 \frac{\beta}{2} = \prod_{n=-\infty}^{\infty} \frac{\beta^2 + 4n^2\pi^2}{\beta^2 + (1+2n)^2\pi^2}$$

or

$$\tanh^2 t = \prod_{n=-\infty}^{\infty} \frac{4(t^2 + n^2\pi^2)}{4t^2 + (1+2n)^2\pi^2} \tag{16}$$

which yields the following expansion for the hyperbolic tangent function

$$\tanh t = \pm \prod_{n=-\infty}^{\infty} 2\sqrt{\frac{t^2 + n^2\pi^2}{4t^2 + (1+2n)^2\pi^2}} \tag{17}$$

where the upper case relation holds for $t \geq 0$, while in the lower case relation t is supposed to be less then zero.

It can readily be shown that the infinite product representations in (17) and (16) uniformly converge for $-\infty < t < \infty$. Proof of the convergence can be accomplished in the way used for the product in (10).

3 Dirichlet-Neumann problem on an infinite strip

Continuing the review of classical Green's functions, we consider a mixed boundary value problem for Laplace equation on the infinite strip $\Omega = \{-\infty < x < \infty,\ 0 < y < b\}$, with Dirichlet condition imposed on $y = 0$, while Neumann condition is imposed on $y = b$. Recall the Green's function for this formulation, which is expressed in [3] as

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \left(\ln \sqrt{\frac{1 + 2e^{\omega(x-\xi)} \cos \omega(y-\eta) + e^{2\omega(x-\xi)}}{1 - 2e^{\omega(x-\xi)} \cos \omega(y-\eta) + e^{2\omega(x-\xi)}}} \right.$$

$$\left. + \ln \sqrt{\frac{1 - 2e^{\omega(x-\xi)} \cos \omega(y+\eta) + e^{2\omega(x-\xi)}}{1 + 2e^{\omega(x-\xi)} \cos \omega(y+\eta) + e^{2\omega(x-\xi)}}} \right), \quad \omega = \frac{\pi}{2b} \tag{18}$$

Following the procedure of method of images described earlier, we look for an alternative to (18) representation of the Green's function. It can be obtained in a form of the aggregate response to an infinite sum of properly spaced unit sources and sinks. Their locations will be chosen in compliance with the following pattern. To compensate the trace of the fundamental solution

$G_0^+(x,y;\xi,\eta)$ on the boundary line $y=0$, a unit sink $S_{1,0}^-$ is placed at the point $(\xi,-\eta)$, with the response given as

$$G_{1,0}^-(x,y;\xi,-\eta)=\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y+\eta)^2}$$

The Neumann condition on $y=b$ can be supported by placing a unit source $S_{1,b}^+$ at the point $(\xi,2b-\eta)$. This yields

$$G_{1,b}^+(x,y;\xi,2b-\eta)=-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(2b-\eta))^2}$$

The trace of $S_{1,b}^+$ on the boundary lines $y=0$ can, in turn, be compensated with a unit sink $S_{2,0}^-$ placed at $(\xi,-2b+\eta)$, with the response given as

$$G_{2,0}^-(x,y;\xi,-2b+\eta)=\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(-2b+\eta))^2}$$

while the Neumann condition on $y=b$ can be supported with a unit sink $S_{2,b}^-$ located at $(\xi,2b+\eta)$, with the response

$$G_{2,b}^-(x,y;\xi,2b+\eta)=\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(2b+\eta))^2}$$

The trace of the sink $S_{2,b}^-$ on $y=0$ can be compensated with a unit source $S_{3,0}^+$ placed at $(\xi,-2b-\eta)$, with the response

$$G_{3,0}^+(x,y;\xi,-2b-\eta)=-\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y+(2b+\eta))^2}$$

while the Neumann condition on $y=b$ can be supported with a unit sink $S_{3,b}^-$ at $(\xi,4b-\eta)$, with the response

$$G_{3,b}^-(x,y;\xi,4b-\eta)=\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-(4b-\eta))^2}$$

Proceeding in compliance with this scheme, the Green's function that we are looking for is obtained in the infinite product form

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta+4nb)^2}{(x-\xi)^2+(y-\eta+4nb)^2}}$$
$$\times\sqrt{\frac{(x-\xi)^2+(y-\eta+2(2n+1)b)^2}{(x-\xi)^2+(y+\eta+2(2n+1)b)^2}} \tag{19}$$

which is an alternative to the closed analytic form in (18).

By comparison of the expressions in (19) and (18), obtained for the same Green's function, one arrives at the identity

$$\prod_{n=-\infty}^{\infty} \frac{(x-\xi)^2+(y+\eta+4nb)^2}{(x-\xi)^2+(y-\eta+4nb)^2}$$

$$\times \prod_{n=-\infty}^{\infty} \frac{(x-\xi)^2+(y-\eta+2(2n+1)b)^2}{(x-\xi)^2+(y+\eta+2(2n+1)b)^2}$$

$$= \frac{1+2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}$$

$$\times \frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1+2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}$$

To obtain a more compact form for this relation, we assume $b=\pi/2$, which evidently implies that $\omega=1$, and introduce the parameters $\beta=x-\xi,\ u=y+\eta$ and $v=y-\eta$. This yields the following identity

$$\prod_{n=-\infty}^{\infty} \frac{[\beta^2+(u+2n\pi)^2]\,[\beta^2+(v+(2n+1)\pi)^2]}{[\beta^2+(v+2n\pi)^2]\,[\beta^2+(u+(2n+1)\pi)^2]}$$

$$= \frac{(1-2e^{\beta}\cos u+e^{2\beta})(1+2e^{\beta}\cos v+e^{2\beta})}{(1-2e^{\beta}\cos v+e^{2\beta})(1+2e^{\beta}\cos u+e^{2\beta})} \tag{20}$$

A particular case of the above identity can be obtained if we assume, for example, $\beta=0$. This reduces the relation in (20) to

$$\prod_{n=-\infty}^{\infty} \frac{(u+2n\pi)^2[v+(2n+1)\pi]^2}{(v+2n\pi)^2[u+(2n+1)\pi]^2}$$

$$= \frac{(1-\cos u)(1+\cos v)}{(1-\cos v)(1+\cos u)} = \tan^2\frac{u}{2}\cot^2\frac{v}{2}$$

It is evident that the above relation holds if

$$\tan\frac{u}{2}\cot\frac{v}{2} = \prod_{n=-\infty}^{\infty} \frac{(u+2n\pi)[v+(2n+1)\pi]}{(v+2n\pi)[u+(2n+1)\pi]} \tag{21}$$

Two infinite product representations of single-variable trigonometric functions can be derived from the above relation. Indeed, if we let $v=\pi/2$ and make the substitution $u/2=t$, then (21) transforms into

$$\tan t = \prod_{n=-\infty}^{\infty} \frac{2(3+4n)(t+n\pi)}{(1+4n)[2t+(2n+1)\pi]} \tag{22}$$

or

$$\tan t = \prod_{n=-\infty}^{\infty} \left\{ 1 + \frac{4t - \pi}{(1+4n)[2t + (2n+1)\pi]} \right\}$$

which converges for any value of the variable t in the domain of the tangent function. This assertion can be proved in the same way as that used in Section 2. The data in Table 2 give a clear view of the convergence rate of the representation in (22).

It is evident that the relation in (22) yields the following infinite product representation

$$\cot t = \prod_{n=-\infty}^{\infty} \frac{(1+4n)[2t + (2n+1)\pi]}{2(3+4n)(t+n\pi)} \tag{23}$$

or

$$\cot t = \prod_{n=-\infty}^{\infty} \left[1 + \frac{\pi - 4t}{2(3+4n)(t+n\pi)} \right]$$

for the cotangent function. By the way, the representation in (23) can be directly obtained from that in (21) by letting $u = \pi/2$ and making the substitution $v/2 = t$. The expansion in (23) uniformly converges for $t \neq n\pi$, where $n = 0,\ \pm 1,\ \pm 2,\ \ldots$.

Another infinite product representation for an elementary function can be obtained from the relation in (20). Indeed, assuming $u = 0$ and $v = \pi$, one obtains

$$\left(\frac{1 - e^{\beta}}{1 + e^{\beta}} \right)^4 = \prod_{n=-\infty}^{\infty} \frac{(\beta^2 + 4n^2\pi^2)[\beta^2 + 4(1+n)^2\pi^2]}{[\beta^2 + (1+2n)^2\pi^2]^2}$$

which converts to the infinite product expansion

$$\tanh^4 \frac{\beta}{2} = \prod_{n=-\infty}^{\infty} \frac{(\beta^2 + 4n^2\pi^2)[\beta^2 + 4(1+n)^2\pi^2]}{[\beta^2 + (1+2n)^2\pi^2]^2}$$

of the fourth power of the hyperbolic tangent. The above can be rewritten as

$$\tanh^4 t = \prod_{n=-\infty}^{\infty} \frac{16(t^2 + n^2\pi^2)[t^2 + (1+n)^2\pi^2]}{[4t^2 + (1+2n)^2\pi^2]^2}$$

$$= \prod_{n=-\infty}^{\infty} \frac{16(t^2 + n^2\pi^2)^2}{[4t^2 + (1+2n)^2\pi^2]^2} \tag{24}$$

with the equivalent form

$$\tanh^4 t = \prod_{n=-\infty}^{\infty} \left\{ 1 + \frac{\pi^2[8(t^2 - n(n+1)\pi^2) - \pi^2]}{[4t^2 + (1+2n)^2\pi^2]^2} \right\}$$

where the expansion uniformly converges for any value of t.

It is evident that the expansion in (24) can directly be obtained from that derived for $\tanh^2 t$ in (16).

4 Problems on a semi-infinite strip

In this section, we apply our technique to boundary value problems formulated for Laplace equation on the semi-infinite strip $\Omega = \{0 < x < \infty,\ 0 < y < b\}$. For the first example, consider Dirichlet problem whose classical compact form

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\left(\ln\sqrt{\frac{1-2e^{\omega(x+\xi)}\cos\omega(y-\eta)+e^{2\omega(x+\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}}\right.$$
$$\left.+\ln\sqrt{\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x+\xi)}\cos\omega(y+\eta)+e^{2\omega(x+\xi)}}}\right), \qquad \omega = \frac{\pi}{b} \tag{25}$$

of the Green's function is well-known in literature. In [3], for example, it is obtained by the modified version of the method of eigenfunction expansion.

Tracing out the procedure of the method of images described earlier in detail, one arrives at an alternative to that in (25) infinite product representation

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}}$$
$$\times\sqrt{\frac{(x+\xi)^2+(y-\eta+2nb)^2}{(x+\xi)^2+(y+\eta-2nb)^2}} \tag{26}$$

for the Green's function. The alternative expressions in (26) and (25) give a rise to the following identity

$$\prod_{n=-\infty}^{\infty}\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}$$
$$\times\frac{(x+\xi)^2+(y-\eta+2nb)^2}{(x+\xi)^2+(y+\eta-2nb)^2}$$
$$=\frac{1-2e^{\omega(x+\xi)}\cos\omega(y-\eta)+e^{2\omega(x+\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}$$
$$\times\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x+\xi)}\cos\omega(y+\eta)+e^{2\omega(x+\xi)}}$$

To view this identity in a more compact form, we assume $b = \pi$ and introduce the parameters $\alpha = x+\xi,\ \ \beta = x-\xi,\ \ u = y+\eta$ and $v = y-\eta$. This reduces the above identity to

$$\prod_{n=-\infty}^{\infty}\frac{[\beta^2+(u-2n\pi)^2]\,[\alpha^2+(v+2n\pi)^2]}{[\beta^2+(v+2n\pi)^2]\,[\alpha^2+(u-2n\pi)^2]}$$

$$= \frac{(1-2e^{\alpha}\cos v+e^{2\alpha})(1-2e^{\beta}\cos u+e^{2\beta})}{(1-2e^{\beta}\cos v+e^{2\beta})(1-2e^{\alpha}\cos u+e^{2\alpha})} \tag{27}$$

which reads as

$$\coth^2\frac{\alpha}{2}\tanh^2\frac{\beta}{2} = \prod_{n=-\infty}^{\infty}\frac{(\beta^2+4n^2\pi^2)\left[\alpha^2+(1+2n)^2\pi^2\right]}{(\alpha^2+4n^2\pi^2)\left[\beta^2+(1+2n)^2\pi^2\right]} \tag{28}$$

if one assigns the values $u = 0$ and $v = \pi$ to the variables in (27).

It is worth noting that the expansion in (16) (see Section 2) follows from that in (28) as α approaches infinity. If, on the other hand, the limit is taken in (28) as β goes to infinity, then one arrives at

$$\coth^2\frac{\alpha}{2} = \prod_{n=-\infty}^{\infty}\frac{\alpha^2+(1+2n)^2\pi^2}{\alpha^2+4n^2\pi^2}$$

or

$$\coth^2 t = \prod_{n=-\infty}^{\infty}\frac{4t^2+(1+2n)^2\pi^2}{4(t^2+n^2\pi^2)} \tag{29}$$

from which an expansion for the hyperbolic cotangent directly follows as

$$\coth t = \pm\prod_{n=-\infty}^{\infty}\frac{1}{2}\sqrt{\frac{4t^2+(1+2n)^2\pi^2}{t^2+n^2\pi^2}} \tag{30}$$

with the upper case holding if $t \geq 0$, while in the lower case $t < 0$.

The uniform convergence of the expansions in (29) and (30) is evident for any non-zero value of t. And it is also evident that the above expansions for $\coth^2 t$ and $\coth t$ can directly be obtained from those in (16) and (17).

For another example, we consider a mixed boundary value problem stated on the semi-infinite strip $\Omega = \{0 < x < \infty,\ 0 < y < b\}$. Let Dirichlet conditions be imposed on the boundary fragments $y = 0$ and $y = b$, while Neumann condition be imposed on $x = 0$. The compact form Green's function

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\left(\ln\sqrt{\frac{1-2e^{\omega(x+\xi)}\cos\omega(y+\eta)+e^{2\omega(x+\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}}\right.$$

$$\left.+\ln\sqrt{\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x+\xi)}\cos\omega(y-\eta)+e^{2\omega(x+\xi)}}}\right), \qquad \omega = \frac{\pi}{b} \tag{31}$$

is obtained for this problem in [3] by the modified version of the method of eigenfunction expansion. Following the procedure of the method of images, one arrives at an alternative to that in (31) infinite product representation

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}}$$

$$\times\sqrt{\frac{(x+\xi)^2+(y+\eta-2nb)^2}{(x+\xi)^2+(y-\eta+2nb)^2}} \tag{32}$$

for the Green's function. Setting equal the arguments of the logarithmic functions in (31) and (32), one immediately obtains

$$\prod_{n=-\infty}^{\infty}\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}$$

$$\times\frac{(x+\xi)^2+(y+\eta-2nb)^2}{(x+\xi)^2+(y-\eta+2nb)^2}$$

$$=\frac{1-2e^{\omega(x+\xi)}\cos\omega(y+\eta)+e^{2\omega(x+\xi)}}{1-2e^{\omega(x-\xi)}\cos\omega(y-\eta)+e^{2\omega(x-\xi)}}$$

$$\times\frac{1-2e^{\omega(x-\xi)}\cos\omega(y+\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x+\xi)}\cos\omega(y-\eta)+e^{2\omega(x+\xi)}}$$

Similarly to the case of the Dirichlet problem, we assume $b=\pi$ and introduce the parameters $\alpha=x+\xi,\ \ \beta=x-\xi,\ \ u=y+\eta$ and $v=y-\eta$. This yields

$$\prod_{n=-\infty}^{\infty}\frac{[\beta^2+(u-2n\pi)^2]\,[\alpha^2+(u-2n\pi)^2]}{[\beta^2+(v+2n\pi)^2]\,[\alpha^2+(v+2n\pi)^2]}$$

$$=\frac{(1-2e^{\alpha}\cos u+e^{2\alpha})(1-2e^{\beta}\cos u+e^{2\beta})}{(1-2e^{\beta}\cos v+e^{2\beta})(1-2e^{\alpha}\cos v+e^{2\alpha})}$$

from which the relation

$$\frac{(1+e^{\alpha})^2(1+e^{\beta})^2}{(1+e^{2\alpha})(1+e^{2\beta})}$$

$$=\prod_{n=-\infty}^{\infty}\frac{16[\alpha^2+(1-2n)^2\pi^2]\,[\beta^2+(1-2n)^2\pi^2]}{[4\alpha^2+(1+4n)^2\pi^2]\,[4\beta^2+(1+4n)^2\pi^2]}$$

follows if it is assumed that $u=\pi$ and $v=\pi/2$.

Due to the symmetry of the right-hand side and the left-hand side in the above relation with respect to the parameters α and β, we can state

$$\frac{(1+e^{t})^2}{1+e^{2t}}=\prod_{n=-\infty}^{\infty}\frac{4[t^2+(1-2n)^2\pi^2]}{4t^2+(1+4n)^2\pi^2}$$

or, converting to a form of the hyperbolic functions

$$\frac{1+\cosh t}{\cosh t}=\prod_{n=-\infty}^{\infty}\frac{4[t^2+(1-2n)^2\pi^2]}{4t^2+(1+4n)^2\pi^2} \tag{33}$$

This leads to the product expansion

$$\operatorname{sech} t = -1 + \prod_{n=-\infty}^{\infty} \frac{4[t^2 + (1-2n)^2\pi^2]}{4t^2 + (1+4n)^2\pi^2}$$

of the hyperbolic secant, which uniformly converges for any value of t. Equivalent form for the above reads

$$\operatorname{sech} t = -1 + \prod_{n=-\infty}^{\infty} \left[1 + \frac{3(1-8n)\pi^2}{4t^2 + (1+4n)^2\pi^2}\right] \tag{34}$$

Table 3 brings some data that illustrate the practical convergence of the expansion in (34).

Table 1: Relative error (%) of computing values of $\sin t$ by the K-th partial product of the expansions in (12) and (13).

Eqn. No.	(12)			(13)		
K	10	20	80	10	20	80
$t = \pi/4$	1.25	0.64	0.16	0.42	0.22	0.05
$t = \pi/2$	0.00	0.00	0.00	2.41	1.23	0.31
$t = 3\pi/4$	2.13	1.09	0.27	3.89	1.97	0.50

Table 2: Relative error (%) of computing values of $\tan t$ by the K-th partial product of the expansion in (22).

t	$\pi/6$	$\pi/4$	$\pi/3$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$K = 10$	0.79	0.00	0.79	3.91	4.67	5.43
$K = 20$	0.41	0.00	0.41	2.01	2.40	2.81
$K = 80$	0.10	0.00	0.10	0.51	0.62	0.72

Table 3: Relative error (%) of computing values of sech$t$ by the K-th partial product of the expansions in (34).

t	0.0	2.0	4.0	6.0	8.0	10.0
$K = 10$	7.27	4.59	3.75	3.62	3.58	3.55
$K = 20$	3.69	2.33	1.91	1.85	1.84	1.83
$K = 80$	0.93	0.59	0.48	0.47	0.47	0.47

Closure

The method of images was used to obtain alternative representations for some classical Green's functions to boundary value problems for Laplace equation in two dimensions. This gives a rise to special identities involving infinite products. The latter, in turn, allow to derive new infinite product expansions for a number of elementary functions.

To the best of the author's knowledge, the infinite product representations of elementary functions, derived in this study, have never been published before and are obtained for the first time. It is evident that the work can make a notable contribution to various areas of applied mathematical analysis. And the area where the input of this study is expected to be seen really soon is the approximation of functions. It will also be of interest to researchers in other areas of applied mathematics.

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