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Asymptotic Compensation in Discrete Distributed Systems: Analysis, Approximations and Simulations

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Abstract

This paper concerns the problem of asymptotic compensation for a class of discrete disturbed systems. It is an extension of previous works on the problem of finite time space compensation studied for continuous linear systems. In this work, we introduce and we characterize the asymptotic exact and weak remediability and we study their relationship with the asymptotic notions of controllability, stability and stabilizability.

The minimum energy problem is studied with an extension to the case where the observation is affected by a measurement error. The cases of multi-actuators and multi-sensors are considered. An application with illustrative examples is given and various other situations are examined. Approximations and numerical results are also presented.

Keywords: Discrete distributed systems, asymptotic remediability, control, observation, actuators, sensors

1 Introduction

In this paper which concerns the asymptotic analysis of a class of discrete linear distributed systems, we study the possibility to compensate asymptotically the effect of a known or unknown disturbance. This problem, particularly motivated by pollution problems and so-called space compensation or remediability problem, has been studied in previous works [1,2,3,4] for finite time continuous systems. It consists to study, with respect to the observation, the possibility of finite time compensation of the effect of a known or unknown disturbance. In these works, various systems and situations are considered and studied.

One know the great importance of the asymptotic analysis in systems theory. It is then natural to consider an extension to the asymptotic discrete case of this problem. In this paper, we give characterization results and we show that also in the asymptotic discrete case, the remediability remain weaker than the controllability. The corresponding minimum energy problem is examined and the relationship with the notions of stability and stabilizability is studied. We particularly show that a discrete system can be asymptotically remediable without being stable or stabilizable. Various other problems and situations are considered, illustrative examples and numerical results are also presented.

This paper is organized as follows:

In the second paragraph, we introduce the notions of weak and exact remediability in the case of a finite time horizon.

In the third paragraph, and under convenient hypothesis, we define and we characterize the notions of weak and exact asymptotic remediability. We show how to find an input operator with respect to the output one ensuring the asymptotic compensation of any disturbance on the system. The cases of multi-sensors and multi-actuators are also examined. Then, we introduce and we characterize the notion of asymptotically efficient actuators.

The paragraph 4 is consecrated to the problem of asymptotic remediability with minimal energy in the case where the observation is exact or affected by a measurement error. We show how to find the optimal control ensuring the asymptotic compensation of a disturbance, and then, we characterize the set of disturbances which are exactly remediable asymptotically.

In the fifth paragraph, we define the notions of asymptotic weak and exact controllabilty and we give their characterization. We study the relationship between the asymptotic remediability and the asymptotic controllability, and hence between asymptotic strategic actuators and asymptotic efficient actuators.

Then, we examine by the same the nature of its relation with the notions of stability and stabilizability. Hence, we show that a discrete linear system can be remediable asymptotically but not stable or even stabilizable. Other situations are also considered.

In paragraph 6, we examine the case of a discrete version of a diffusion process with a Dirichlet or Neuman boundary conditions.

In the last paragraph, we present illustrative examples as well as approximations and numerical results.

2 Finite time case

We consider, without loss of generality, a class of linear distributed systems described by the following discrete equation:

$$(S_d)\begin{cases} z_{k+1} = \phi z_k + Bu_k + f_k \quad ; 0 \leq k \leq N-1 \\ z_0 \in Z \end{cases} \tag{1}$$

where $\phi \in \mathcal{L}(Z)$; $B \in \mathcal{L}(U, Z)$; z_k, $f_k \in Z$ and $u_k \in U$ are respectively the state, the disturbance, the control at step k; N is an integer, $N \geq 1$; Z and U are supposed to be Hilbert spaces.

The system (S_d) is augmented by the output equation :

$$(E_d) \;\; y = Cz \tag{2}$$

where $C \in \mathcal{L}(Z, Y)$, $z = (z_1, ..., z_N)^{tr}$, $y = (y_1, ..., y_N)^{tr}$ with $y_k = Cz_k$ for $k = 1, ..., N$; Y is the observation space, a Hilbert space.

Let $f^{(k)} = (f_0, ..., f_{k-1})^{tr}$, $u^{(k)} = (u_0, ..., u_{k-1})^{tr}$, $f = f^{(N)}$ and $u = u^{(N)}$. The state of system (S_d) and the observation at step N are respectively given by:

$$z_N = \phi^N z_0 + \sum_{i=0}^{N-1} \phi^{N-1-i} f_i + \sum_{i=0}^{N-1} \phi^{N-1-i} Bu_i \tag{3}$$

and

$$y_N = C\phi^N z_0 + \sum_{i=0}^{N-1} C\phi^{N-1-i} Bu_i + \sum_{i=0}^{N-1} C\phi^{N-1-i} f_i \tag{4}$$

In the case without disturbance and without control, the observation at the final step is given by:

$$y_N = C\phi^N z_0$$

But if $u \neq 0$ and $f \neq 0$, generally $y_N \neq C\phi^N z_0$.

The problem consists to study the existence of an input operator, with respect to the output one, ensuring at the last step, the compensation of any disturbance on the system, i.e.

For any $f = (f_0, f_1, ..., f_{N-1}) \in Z^N$, there exists $u = (u_0, u_1, ..., u_{N-1}) \in U^N$ such that:

$$\sum_{i=0}^{N-1} C\phi^{N-1-i} B u_i + \sum_{i=0}^{N-1} C\phi^{N-1-i} f_i = 0 \tag{5}$$

Let us consider the following operators:

$$\begin{array}{rcccl} H_N & : & U^N & \longrightarrow & Z \\ & & u & \longrightarrow & H_N u = \displaystyle\sum_{i=0}^{N-1} \phi^{N-1-i} B u_i \end{array} \tag{6}$$

$$\begin{array}{rcccl} \overline{H}_N & : & Z^N & \longrightarrow & Z \\ & & f & \longrightarrow & \overline{H}_N f = \displaystyle\sum_{i=0}^{N-1} \phi^{N-1-i} f_i \end{array} \tag{7}$$

The equality (5) becomes

$$CH_N u + R_N f = 0 \tag{8}$$

with

$$R_N = C\overline{H}_N \tag{9}$$

This leads to the following definitions:

Definition 2.1 $(S_d) + (E_d)$ *is said to be*

i) exactly remediable, if for any $f \in Z^N$, *there exists* $u \in U^N$ *such that*

$$CH_N u + R_N f = 0$$

ii) weakly remediable, if for any $f \in Z^N$ *and any* $\varepsilon > 0$, *there exists* $u \in U^N$ *such that:*

$$\|CH_N u + R_N f\| < \varepsilon$$

The characterization results are similar to those established in the finite time case for continuous linear systems [1,2,3,4]

3 Asymptotic case

3.1 Problem statement

In this part, we consider the following system, also noted (S_d)

$$(S_d)\begin{cases} z_{k+1} = \phi z_k + Bu_k + f_k \\ z_0 \in Z \quad ; \qquad k \geq 0 \end{cases} \tag{10}$$

augmented by the output equation, also noted (E_d)

$$(E_d) \;\; y_k = Cz_k \; ; \quad \text{k} \geq 0 \tag{11}$$

and the operators H^∞ and $\overline{H}^\infty$ defined by:

$$\begin{array}{llcl} H^\infty : & \ell^2(U) & \longrightarrow & Z \\ & u = (u_0, ..., u_N, ...)^{tr} & \longrightarrow & H^\infty u = \displaystyle\sum_{k=0}^{+\infty} \phi^k B u_k \end{array} \tag{12}$$

$$\begin{array}{llcl} \overline{H^\infty} : & \ell^2(Z) & \longrightarrow & Z \\ & f = (f_0, ..., f_N, ...)^{tr} & \longrightarrow & \overline{H^\infty} f = \displaystyle\sum_{k=0}^{+\infty} \phi^k f_k \end{array} \tag{13}$$

and R^∞ defined by

$$R^\infty = C\overline{H^\infty} \tag{14}$$

The problem consists to study, with respect to the output of the system, the existence of an input operator B ensuring the asymptotic compensation of any disturbance, i.e.

For any $f = (f_0, ..., f_N, ...)^{tr} \in \ell^2(Z)$, there exists $u = (u_0, ..., u_N, ...)^{tr} \in \ell^2(U)$ such that

$$CH^\infty u + R^\infty f = 0 \tag{15}$$

Let us note that if

$$(\|\phi^k\|)_{k\geq 0} \in \ell^2(I\!R) \tag{16}$$

then the operators H^{∞} and $\overline{H}^{\infty}$ are well defined, this is also the case for CH^{∞} and R^{∞}. Hence, if ϕ is contraction operator, i.e.

$$\| \phi \| < 1$$

or if (S_d) is exponentially stable:

$$\| \phi^k \| \leq e^{-\alpha k}; \quad k \geq 1 \quad (\alpha > 0)$$

then (16) is satisfied. In fact, we are concerned by the operators H_C^{∞} and R_C^{∞} defined by

$$H_C^{\infty} u \equiv \sum_{k=o}^{+\infty} C\phi^k B u_k = CH^{\infty} u$$

and

$$R_C^{\infty} f \equiv \sum_{k=0}^{+\infty} C\phi^k f_k$$

then one can consider a weaker hypothesis. Indeed, we assume that

$$(\left\| C\phi^k \right\|)_{k\geq 0} \in \ell^2(I\!R) \tag{17}$$

In this case, H_C^{∞} and R_C^{∞} are well defined and (15) becomes

$$H_C^{\infty} u + R_C^{\infty} f = 0 \tag{18}$$

Under hypothesis (17), the notions of weak and exact asymptotic remediability can be formulated as follows:

Definition 3.1 *We say that*
1)(S_d)+(E_d) is exactly remediable asymptotically if for any $f = (f_k)_{k\geq 0} \in \ell^2(Z)$, there exists $u = (u_k)_{k\geq 0} \in \ell^2(U)$ such that:

$$H_C^{\infty} u + R_C^{\infty} f = 0$$

2) $(S_d)+(E_d)$ is weakly remediable asymptotically, if for any $f = (f_k)_{k\geq 0} \in \ell^2(Z)$ and any $\varepsilon > 0$, there exists $u = (u_k)_{k\geq 0} \in \ell^2(U)$ such that:

$$\| H_C^{\infty} u + R_C^{\infty} f \| < \varepsilon$$

Let us note that for an integer $M > 1$, the operators H^M and R^M defined by:

$$H^M = \sum_{k=0}^{M-1} \phi^k B u_k$$

and

$$R^M f = \sum_{k=0}^{M-1} C\phi^k f_k$$

are such that:

$$H^M u = H_M v \text{ and } R^M f = R_M g$$

where v and g are defined by $v_k = u_{M-1-k}$ and $g_k = f_{M-1-k}$; H_M and R_M are given respectively by (6) and (9). Moreover, we have

$$\begin{aligned} H_C^\infty u + R_C^\infty f &= CH^M u + R^M f + \sum_{k=M}^{+\infty} C\phi^k B u_k + \sum_{k=M}^{+\infty} C\phi^k \mathrm{f}_k \\ &= [CH_M v + R_M g] + [\epsilon_1(M) + \epsilon_2(M)] \end{aligned} \tag{19}$$

with

$$\lim_{M\to+\infty} [\epsilon_1(M) + \epsilon_2(M)] = 0$$

then

$$\lim_{M\to+\infty} [CH_M v + R_M g] = H_C^\infty u + R_C^\infty f$$

3.2 Characterization

First, let us remark that for $f_k = -Bu_k$, we have $R_C^\infty f = -H_C^\infty u$, then

$$Im(H_C^\infty) \subset Im(R_C^\infty) \tag{20}$$

We have the following characterization result where in the general case, P^* and W' are respectively the adjoint operator of P and the dual space of W.

Proposition 3.2 *Under hypothesis (17):*

(i) $(S_d) + (E_d)$ *is exactly remediable asymptotically if and only if*

$$Im(R_C^\infty) \subset Im(H_C^\infty)$$

this is equivalent to :

(ii) $\exists \gamma > 0$ *such that:*

$$\sum_{k=0}^{+\infty} \left\|(\phi^*)^k C^* \theta\right\|_{Z'}^2 \leq \gamma \sum_{k=0}^{+\infty} \left\|B^*(\phi^*)^k C^* \theta\right\|_{U'}^2 \ ; \ \forall \theta \in Y^{'} \tag{21}$$

Proof:
(i) derives from the definition.
(ii) derives from the following lemma [5,6,7]

Lemma 3.3 *Let* X, Y, Z *be reflexive Banach spaces,* $P \in \mathcal{L}(X, Z)$ *and* $Q \in \mathcal{L}(Y, Z)$*. There is equivalence between:*

$$Im(P) \subset Im(Q)$$

and

$\exists \gamma > 0$ *such that for any* $z^* \in Z'$, *we have* $\|P^* z^*\|_{X'} \leq \gamma \|Q^* z^*\|_{Y'}$

Concerning the weak asymptotic remediability, we have the following result.

Proposition 3.4 *Under hypothesis (17):*

(i) $(S_d) + (E_d)$ *is weakly remediable asymptotically if and only if*

$$Im(R_C^\infty) \subset \overline{Im(H_C^\infty)} \tag{22}$$

this is equivalent to

(ii)

$$Ker[B^*(R_C^\infty)^*] = Ker[(R_C^\infty)^*] \tag{23}$$

or

$$\bigcap_{k \geq 0} Ker\left[B^*(\phi^*)^k C^*\right] = Ker\left[C^*\right] \tag{24}$$

Proof: (i) derives immediately from the definition.

(i) equivalent to (ii) derives from (22) by considering the orthogonal space and using (20) to deduce that:

$$Ker[(R_C^\infty)^*] = Ker[(H_C^\infty)^*]$$

Since

$$(R_C^\infty B)(u) = R_C^\infty (Bu) = \sum_{k\geq 0} C\phi^k B u_k = H_C^\infty u$$

then $R_C^\infty B = H_C^\infty$ and hence $(H_C^\infty)^* = B^*(R_C^\infty)^*$, we then have the result.

On the other hand, we have $(R_C^\infty)^* = (C^*, \phi^* C^*, ..., (\phi^*)^k C^*, ...)^{tr}$, then $Ker(C^*) = Ker\left[(R_C^\infty)^*\right]$.

It is easy to show that:

$$Ker\left[B^*(R_C^\infty)^*\right] = \bigcap_{k\geq 0} Ker\left[B^*(\phi^*)^k C^*\right]$$

then we have (24). □

We examine hereafter the case where the system is excited by actuators and where the output is given by sensors [7].

3.3 Case of multi-actuators and multi-sensors

In the case of p actuators $(\Omega_i, g_i)_{1\leq i\leq p}$, we have $U = I\!R^p$, $Z = L^2(\Omega)$ and

$$\begin{array}{rccl} B: & I\!R^p & \longrightarrow & L^2(\Omega) \\ & u_k & \longrightarrow & Bu_k = \displaystyle\sum_{i=1}^{p} g_i u_k^i \end{array} \tag{25}$$

where $u_k = (u_k^1, \cdots, u_k^p)^{tr} \in I\!R^p$, $g_i \in L^2(\Omega_i)$; $\Omega_i = supp(g_i) \subset \Omega$.

We have

$$B^* z = (\langle g_1, z\rangle, \cdots, \langle g_p, z\rangle)^{tr} \tag{26}$$

In this case, the characterization results becomes:

Proposition 3.5 *$(S_d) + (E_d)$ is exactly remediable asymptotically if and only if, there exists $\gamma > 0$ such that:*

$$\sum_{k \geq 0} \left\| (\phi^*)^k C^* \theta \right\|^2_{Z'} \leq \gamma \sum_{i=1}^{p} \sum_{k \geq 0} (\langle \phi^k g_i, C^* \theta \rangle)^2 ; \forall \theta \in Y' \tag{27}$$

Proof: derives from (21) and (26). □

Moreover, if the output is given by q zone sensors $(D_i, h_i)_{1 \leq i \leq q}$, where $h_i \in L^2(D_i)$; $D_i = supp(h_i)$ and for $i \neq j$, $D_i \cap D_j = \emptyset$, the operator C is defined by:

$$\begin{array}{rcll} C: & L^2(\Omega) & \longrightarrow & I\!R^q \\ & z & \longrightarrow & Cz = (\langle h_1, z \rangle, \cdots, \langle h_q, z \rangle)^{tr} \end{array} \tag{28}$$

and its adjoint by

$$C^* \theta = \sum_{i=1}^{q} \theta_i h_i \text{ for } \theta = (\theta_1, ..., \theta_q) \in I\!R^q \tag{29}$$

We also have the following characterization result.

Corollary 3.6 *$(S_d) + (E_d)$ is exactly remediable asymptotically if and only if, there exists $\gamma > 0$ such that $\forall \; \theta = (\theta_1, \cdot\cdot, \theta_q) \in I\!R^q$:*

$$\sum_{k \geq 0} \left\| \sum_{j=1}^{q} \theta_j (\phi^*)^k h_j \right\|^2_{Z'} \leq \gamma \sum_{k \geq 0} \sum_{i=1}^{p} \left(\sum_{j=1}^{q} \theta_j \langle \phi^k g_i, h_j \rangle \right)^2$$

In the case of one sensor (D, h) and one actuator (Ω, g), the previous inequality becomes

$$\sum_{k \geq 0} \left\| (\phi^*)^k h \right\|^2_{Z'} \leq \gamma \sum_{k \geq 0} (\langle \phi^k g, h \rangle)^2 \tag{30}$$

3.4 Notion of asymptotic efficient actuators

We introduce hereafter the notion of asymptotic efficient actuators and we give their characterization in the case of a class of linear systems.

Definition 3.7 *Actuators* $(\Omega_i, g_i)_{1\le i\le p}$ *are said to be asymptotically efficient, or just efficient, if the corresponding system* $(S_d)+(E_d)$ *is weakly remediable asymptotically.*

Proposition 3.8 $(\Omega_i, g_i)_{1\le i\le p}$ *are efficient if and only if*

$$\left\langle \phi^k g_i, C^*\theta \right\rangle = 0; \forall k \geq 0, \forall i = 1, ..., p \implies C^*\theta = 0$$

Proof: Derives from (24) and the fact that:

$$B^*(\phi^*)^k C^*\theta = \left(\left\langle \phi^k g_1, C^*\theta \right\rangle, ..., \left\langle \phi^k g_p, C^*\theta \right\rangle\right)^{tr} \tag{31}$$

□

We consider, without loss of generality, the system (S_d) with ϕ defined by:

$$\phi z = \sum_{n\geq 1} e^{\lambda_n \tau} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj} \tag{32}$$

where $\tau > 0$ (generally small); $\lambda_1, \lambda_2, ...$ are real numbers such that $\lambda_1 > \lambda_2 > \lambda_3 > ....$, $\{\varphi_{nj}, n \geq 1; j = 1, ..., r_n\}$ is an orthonormal basis of Z constituted by eigenfunctions of the operator A defined by

$$Az = \sum_{n\geq 1} \lambda_n \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj} \tag{33}$$

r_n is the multiplicity of the eigenvalue λ_n. We suppose that:

$$\sup_{n\geq 1} \lambda_n = \lambda_1 < 0 \tag{34}$$

ϕ is a self-adjoint operator and

$$\phi^k z = \sum_{n\geq 1} e^{k\lambda_n \tau} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj} \tag{35}$$

The system (S_d) can be considered as a discrete version of a diffusion process. It is augmented by the output equation:

$$(E_d)\ \ y_k = Cz_k;\ \ k \geq 0 \tag{36}$$

and the operator B is given by (25) , i.e.

$$Bu_k = \sum_{i=1}^{p} g_i u_k^i$$

where $u_k = (u_k^1, ..., u_k^p)$.

For $n \geq 1$, let f_n be the function defined by

$$\begin{array}{rlll} f_n: & Y^\star & \longrightarrow & I\!R^{r_n} \\ & \theta & \longrightarrow & f_n(\theta) = (< C^\star\theta, \varphi_{n1} >, ..., < C^\star\theta, \varphi_{nr_n} >)^{tr} \end{array} \tag{37}$$

where $Y^\star$ is the dual space of Y and let M_n be the matrix

$$M_n = (\langle g_i, \varphi_{nj}\rangle)_{1\leq i\leq p; 1\leq j\leq r_n} \tag{38}$$

We have the following characterization result.

Proposition 3.9 *Actuators $(\Omega_i, g_i)_{i=1,p}$ are asymptotically efficient if and only if*

$$ker(C^\star) = \bigcap_{n\geq 1} ker(M_n f_n) \tag{39}$$

Proof: $(S_d) + (E_d)$ is weakly remediable asymptotically if and only if

$$ker[B^\star(R_C^\infty)^\star] = ker[(R_C^\infty)^\star] \tag{40}$$

For $\theta \in Y^\star$, we have:

$$(R_C^\infty)^*\theta = 0 \Longleftrightarrow (\phi^*)^k C^*\theta = 0;\ \forall k \geq 0 \Longleftrightarrow C^*\theta = 0$$

Then

$$ker[(R_C^\infty)^\star] = ker(C^*) \tag{41}$$

Moreover, we have

$$B^*(R_C^\infty)^*\theta = 0 \Longleftrightarrow B^*(\phi^*)^k C^*\theta = 0; \quad \forall k \geq 0$$

Indeed

$$\begin{array}{rcl} B^*(R_C^\infty)^*\theta = 0 & \Longleftrightarrow & B^*(\phi^*)^k C^*\theta = 0\ ;\ \forall k \geq 0 \\ & \Longleftrightarrow & (\left\langle g_1, (\phi^*)^k C^*\theta \right\rangle, ..., \left\langle g_p, (\phi^*)^k C^*\theta \right\rangle)^{tr} = 0\ ;\ \forall k \geq 0 \\ & \Longleftrightarrow & \left\langle g_i, (\phi^*)^k C^*\theta \right\rangle = 0\ ;\ \forall k \geq 0 \text{ and } i = 1, ..., p \end{array}$$

Since

$$\left\langle g_i, (\phi^*)^k C^*\theta \right\rangle = \sum_{n\geq 1} e^{k\lambda_n \tau} \sum_{j=1}^{r_n} \langle g_i, \varphi_{nj} \rangle \left\langle \varphi_{n_j}, C^*\theta \right\rangle$$

and

$$\sum_{n\geq 1} e^{k\lambda_n \tau} \sum_{j=1}^{r_n} \langle g_i, \varphi_{nj} \rangle \left\langle \varphi_{n_j}, C^*\theta \right\rangle = 0 \text{ for } k \geq 0 \text{ and } \tau > 0$$

is equivalent to

$$\sum_{j=1}^{r_n} \langle g_i, \varphi_{nj} \rangle \left\langle \varphi_{n_j}, C^*\theta \right\rangle = 0\ ; \quad \forall n \geq 1$$

then, we have the result.

□

Now, if the output is given by q zone sensors $(D_i, h_i)_{1\leq i\leq q}$ with $h_i \in L^2(D_i)$, $D_i = supp(h_i) \subset \Omega$ and C is defined by (28), we have the following result.

Proposition 3.10 *Actuators $(\Omega_i, g_i)_{i=1,p}$ are efficient if and only if*

$$\bigcap_{n\geq 1} ker(M_n G_n^{tr}) = \{0\} \tag{42}$$

where G_n is the matrix defined by

$$G_n = (< h_i, \varphi_{nj} >)_{\substack{i=1,q \\ j=1,r_n}}$$

Proof:
Since $D_i \cap D_j = \emptyset$ for $i \neq j$, and measure$(D_i) > 0$, then the functions $(h_i)_{i=1,q}$ are linearly independent and hence

$$ker(C^\star) = \{0\}$$

using (41), we have

$$ker[(R_C^\infty)^\star] = \{0\}$$

On the other hand, we have seen that $B^\star(R_C^\infty)^\star\theta = 0$ is equivalent to

$$\left\langle g_l, (\phi^*)^k C^*\theta \right\rangle = 0 \ ; \ \ \forall k \geq 0 \text{ and } \forall l = 1, ..., p$$

Since

$$\begin{aligned} \left\langle g_l, (\phi^*)^k C^*\theta \right\rangle &= \sum_{n\geq 1} e^{k\lambda_n \tau} \sum_{j=1}^{r_n} < g_l, \varphi_{nj} > \sum_{i=1}^{q} < \varphi_{nj}, C^*\theta > \\ &= \sum_{n\geq 1} e^{k\lambda_n \tau} \sum_{j=1}^{r_n} < g_l, \varphi_{nj} > \sum_{i=1}^{q} \theta_i < \varphi_{nj}, h_i > \end{aligned}$$

we have

$$\left\langle g_l, (\phi^*)^k C^*\theta \right\rangle = 0 \Leftrightarrow \sum_{n\geq 1} e^{k\lambda_n \tau} \sum_{j=1}^{r_n} < g_l, \varphi_{nj} > \sum_{i=1}^{q} \theta_i < \varphi_{nj}, h_i >= 0$$

We obtain by the same

$$\sum_{j=1}^{r_n} < g_l, \varphi_{nj} > \sum_{i=1}^{q} \theta_i < \varphi_{nj}, h_i >= 0; \ \forall n \geq 1, \ \forall l = 1, ..., p$$

i.e.

$$M_n G_n^{tr} \theta = 0; \ \ \forall n \geq 1$$

Consequently

$$ker(B^\star(R_C^\infty)^\star) = \bigcap_{n\geq 1} ker(M_n G_n^{tr})$$

and then we have the result. □

From the previous proposition, it is easy to deduce the following corollary.

Corollary 3.11 *If there exists $n_0 \geq 1$ such that:*

$$rank(M_{n_0} G_{n_0}^{tr}) = q \tag{43}$$

or such that

$$rank(G_{n_0}^{tr}) = q \tag{44}$$

and

$$rank(M_{n_0}) = r_{n_0} \tag{45}$$

then the actuators $(\Omega_i, g_i)_{i=1,p}$ are efficient.

The proof derive immediately from the fact that the conditions (44) and (45) imply (43).

4 Minimum energy problem: The optimal control

Under the condition (17) and the weak asymptotic remediability hypothesis, we study in this section the problem of exact asymptotic compensation with minimal energy in the case where the output is exact or affected by an observation error.

4.1 Case of an observation without error

In this paragraph, we suppose that there exists a control $u \in l^2(U)$ such that:

$$H_C^\infty u + R_C^\infty f = 0 \tag{46}$$

We consider the following optimal control problem:

For $f \in l^2(Z)$, does a minimum energy control $u \in l^2(U)$ satisfying (46)? Let

$$D_d = \left\{ u \in l^2(U) \text{ such that } H_C^\infty u + R_C^\infty f = 0 \right\}$$

D_d is not empty. We consider the following function:

$$J_d(u) = \|H_C^\infty u + R_C^\infty f\|_Y^2 + \|u\|_{l^2(U)}^2$$

The considered problem becomes

$$\min_{u \in D} J_d(u)$$

For its resolution, we will use an extension of the Hilbert Uniqueness Method (H.U.M.). For $\theta \in Y' \equiv Y$, let

$$\|\theta\|_{F_d} = (\sum_{k \geq 0} \left\|B^*(\phi^*)^k C^* \theta\right\|_{U'}^2)^{\frac{1}{2}}$$

$\|.\|_{F_d}$ is a semi-norm. We suppose that it is a norm, this is equivalent to assume that $(S_d) + (E_d)$ is weakly remediable asymptotically. Let us consider

$$F_d = \overline{Y}^{\|.\|_{F_d}}$$

F_d is a Hilbert space with the inner product

$$\langle \theta, \delta \rangle_{F_d} = \sum_{k \geq 0} \left\langle B^*(\phi^*)^k C^* \theta, B^*(\phi^*)^k C^* \delta \right\rangle ; \ \theta, \delta \in F_d$$

We consider the operator Λ_C^∞ defined by

$$\Lambda_C^\infty = H_C^\infty (H_C^\infty)^*$$

We have the following result:

Proposition 4.1 Λ_C^∞ *has a unique extension as an isomorphism* $F_d \to F_d'$ *such that :*

$$\langle \Lambda_C^\infty \theta, \delta \rangle_Y = \langle \theta, \delta \rangle_{F_d} \ \ ; \ \ \forall \theta, \delta \in F_d$$

and

$$\|\Lambda_C^\infty \theta\|_{F_d'} = \|\theta\|_{F_d} \quad ; \forall \theta \in F_d$$

Proof:
Let $\theta \in F_d$. We consider the linear mapping

$$\Lambda_C^\infty \theta : \delta \in Y \longrightarrow \langle \Lambda_C^\infty \theta, \delta \rangle_Y \in I\!R$$

we have

$$\begin{aligned}(\Lambda_C^\infty \theta)(\delta) = \langle \Lambda_C^\infty \theta, \delta \rangle &= \langle \sum_{k \geq 0} C \phi^k B B^* (\phi^*)^k C^* \theta, \delta \rangle \\ &= \sum_{k \geq 0} \langle B^* (\phi^*)^k C^* \theta, B^* (\phi^*)^k C^* \delta \rangle = \langle \theta, \delta \rangle_{F_d}\end{aligned}$$

consequently

$$|(\Lambda_C^\infty \theta)(\delta)| = |\langle \theta, \delta \rangle_{F_d}| \leq \|\theta\|_{F_d} \|\delta\|_{F_d}$$

then $\Lambda_C^\infty \theta$ is continuous on Y for the topology of F_d and can be prolonged continuously and in a unique way to F_d, then $\Lambda_C^\infty \theta \in F_d'$ and we have

$$\langle \Lambda_C^\infty \theta, \delta \rangle_Y = \langle \theta, \delta \rangle_{F_d}; \forall \delta \in Y$$

then $\|\Lambda_C^\infty \theta\|_{F_d'} = \|\theta\|_{F_d}$.

The operator $\Lambda_C^\infty : F_d \longrightarrow F_d'$ is linear. Using the Rietz theorem, it is easy to show that it is surjective.
Λ_C^∞ is also injective. Indeed, for $\theta \in F_d'$ such that $\Lambda_C^\infty \theta = 0$, we have $\langle \Lambda_C^\infty \theta, \theta \rangle = 0$, i.e. $\|\theta\|_{F_d'}^2 = 0$, then $\theta = 0$.
Λ_C^∞ is an isomorphism $F_d \longrightarrow F_d'$. □

We show hereafter how to find the optimal control ensuring the asymptotic compensation of a disturbance f.

Proposition 4.2 *If $R_C^\infty f \in F_d'$, then there exists a unique θ_f in F_d' such that:*

$$\Lambda_C^\infty \theta_f = -R_C^\infty f$$

and the control $u_{\theta_f} = (H_C^\infty)^* \theta_f$ satisfy

$$H_C^\infty u_{\theta_f} + R_C^\infty f = 0$$

Moreover, u_{θ_f} is optimal and

$$\left\| u_{\theta_f} \right\|_{l^2(U)} = \|\theta_f\|_{F'_d}$$

<u>Proof:</u> We have

$$\Lambda_C^\infty \theta_f = \sum_{k\geq 0} C\phi^k BB^*(\phi^*)^k C^* \theta_f = H_C^\infty u_{\theta_f} = -R_C^\infty f$$

D_d is closed, convex and not empty. For $u \in D_d$, we have:

$$J_d(u) = \|u\|^2_{l^2(U)}$$

J_d is strictly convex on D_d, and then has a unique minimum at $u^\star \in D_d$, characterized by:

$$\langle u^\star, v - u^\star \rangle \geq 0; \forall v \in D_d$$

For $v \in D_d$, we have

$$\begin{aligned} \langle u_{\theta_f}, v - u_{\theta_f} \rangle &= \langle (H_C^\infty)^* \theta_f, v - (H_C^\infty)^* \theta_f \rangle \\ &= \langle \theta_f, H_C^\infty v - \Lambda_C^\infty \theta_f \rangle = 0 \end{aligned}$$

Since $u^\star$ is unique, then $u^\star = u_{\theta_f}$ and u_{θ_f} is optimal with

$$\left\| u_{\theta_f} \right\|^2 = \|(H_C^\infty)^* \theta_f\|^2 = \langle \theta_f, \Lambda_C^\infty \theta_f \rangle = \|\theta_f\|^2_{F_d}$$

□

We give hereafter a characterization of the set of disturbances f which are exactly remediable asymptotically. If E_d is the set of such disturbances, i.e.

$$E_d = \left\{ f \in l^2(Z) : \exists u \in l^2(U) \text{ such that } H_C^\infty u + R_C^\infty f = 0 \right\}$$

we have the following result.

Proposition 4.3 *E_d is the reciprocal image of F'_d by R_C^∞, i.e.*

$$R_C^\infty(E_d) = F'_d$$

Proof:
Let $y \in F_d^{'}$, there exists a unique θ in $F_d^{'}$ such that $\Lambda_C^{\infty}\theta = y$, then

$$H_C^{\infty}(H_C^{\infty})^*\theta = y$$

Let u be the control defined by:

$$u = (H_C^{\infty})^*\theta$$

we have $y = H_C^{\infty}u$, and for $f = -Bu \in l^2(Z)$, we have $H_C^{\infty}u = -R_C^{\infty}f = y$, then $y \in R_C^{\infty}(E_d)$.

Conversely, let $y \in R_C^{\infty}(E_d)$, there exists $f \in l^2(Z)$ such that $y = R_C^{\infty}f$ and $H_C^{\infty}u + R_C^{\infty}f = 0$ with $u \in l^2(U)$.

If we identify $H_C^{\infty}u$ with the linear map:

$$L_d: \ \theta \in Y \longrightarrow \langle H_C^{\infty}u, \theta\rangle$$

we have:

$$\begin{aligned} L_d(\theta) &= \langle H_C^{\infty}u, \theta\rangle \\ &= \left\langle \sum_{k\geq 0} C\phi^k B u_k, \theta \right\rangle \\ &= \sum_{k\geq 0} \left\langle u_k, B^*(\phi^*)^k C^*\theta \right\rangle \end{aligned}$$

Then $|L_d(\theta)| \leq \|u\|_{l^2(U)} . \|\theta\|_{F_d}$, consequently L_d is continuous on Y for the topology of F_d and hence can be extended continuously, and in unique way, to the space F_d. Then $L_d \in F_d'$ and $H_C^{\infty}u = -R_C^{\infty}f = -y \in F_d'$, consequently $y \in F_d^{'}$. □

4.2 Case of an observation error

In this part, we assume that the system (S_d) given by (10) is augmented by the following output equation:

$$w = y + e \tag{47}$$

where $y = (y_k)_{k\geq 0}$ is the exact observation given by (11) and $e = (e_k)_{k\geq 0}$ is an observation error generally unknown.

In the normal case where $f = 0$ and $u = 0$, the 'normal' observation $\theta = (\theta_k)_{k\geq 0}$ is given by

$$\theta_k = C\phi^k x_0 + e_k$$

But in the case of a disturbance $f \neq 0$ and without control ($u = 0$), the term corresponding in the observation to the disturbance $f = (f_k)_{k\geq 0}$, is given by

$$R_k f = w_k - \theta_k$$

Moreover, if a control term Bu_k is introduced, the observation becomes

$$\begin{aligned} w_k &= \theta_k + \sum_{i=0}^{k} C\phi^{k-i} f_i + \sum_{i=0}^{k} C\phi^{k-i} B u_i \\ &= \theta_k + R_k f + C H_k u \end{aligned}$$

Then, in this case (i.e. where the observation is not exact), the problem of asymptotic compensation is similar to that considered for an observation without error. It consists to study the existence of an input operator B such that:

$\forall f \in f \in l^2(Z)$: $\exists u \in l^2(U)$ such that the corresponding observation, noted $w(u)$, satisfies the asymptotic condition

$$w_k(u) - \theta_k \longrightarrow 0 \text{ when } k \longrightarrow +\infty$$

With the same notations and as indicated in paragraph 3 about the relation between the finite time and the asymptotic cases, this problem can be formulated as follows:

For $f \in f \in l^2(Z)$, does exists a control $u \in l^2(U)$ such that

$$R_C^{\infty} f + K_C^{\infty} u = 0 \ \ ?$$

The results are similar, under hypothesis (17), and also in this case, the optimal control ensuring the asymptotic compensation of a disturbance $f = (f_k)_{k\geq 0}$ is given by proposition 4.2 .

5 Asymptotic remediability, asymptotic controllability, stability and stabilizability

5.1 Asymptotic remediability and asymptotic controllability

In this part, we introduce the asymptotic controllability (which can be also considered as a stabilizability problem), and we study its relationship with the asymptotic remediability.

We consider the system described by the following equation

$$(S_{0,d})\begin{cases} z_{k+1} &= \phi z_k + Bu_k \ ; \ k \geq 0 \\ z_0 &\in Z \end{cases} \tag{48}$$

We suppose that ϕ satisfy the condition (16). In this case, the operator H^{∞} is well defined.

Definition 5.1 *The system $(S_{0,d})$ is said to be exactly (resp. weakly) controllable asymptotically if*

$$Im(H^{\infty}) = Z \ (\ resp. \ \overline{Im(H^{\infty})} = Z \ )$$

For the exact asymptotic controllability, we have the following result.

Proposition 5.2 *The system $(S_{0,d})$ is exactly controllable asymptotically if and only if*

$$\exists \gamma > 0 \ such\ that \ \parallel z^{\star} \parallel_{Z'} \leq \gamma \left[\sum_{k \geq 0} \left\| B^{*}(\phi^{*})^{k} z^{*} \right\|_{U'}^{2} \right]^{\frac{1}{2}} \ ; \ \forall z^{\star} \in Z'$$

The proof derive from lemma 3.3.

Concerning the weak asymptotic controllability, we have the following characterization.

Proposition 5.3 *There is equivalence between*

i) The system $(S_{0,d})$ is weakly controllable asymptotically

ii)

$$Ker\left[(H^{\infty})^{*}\right] = \{0\}$$

iii)

$$\Lambda^{\infty} = H^{\infty}(H^{\infty})^{\star} \ is\ positive\ definite$$

Proof:
i) equivalent to ii) derives from the definition and from the fact that:

$$\overline{ImH^{\infty}} = [Ker(H^{\infty})^{\star}]^{\perp}$$

The equivalence between ii) and iii) derive from the fact that

$$< \Lambda^{\infty}\theta, \theta >=< H^{\infty}(H^{\infty})^{\star}\theta, \theta >=\| (H^{\infty})^{\star}\theta \|^2$$

then we have

$$< \Lambda^{\infty}\theta, \theta >= 0 \text{ if and only if } \theta \in Ker[(H^{\infty})^{\star}]$$

□

We show hereafter that the remediability remain weaker than the controllability in the asymptotic case.

Proposition 5.4 *If $(S_{0,d})$ is exactly controllable asymptotically, then $(S_d) + (E_d)$ is exactly remediable asymptotically.*

Proof: For $\theta \in Y'$, we have

$$\begin{aligned} \sum_{k\geq 0} \| (\phi^*)^k C^{\star}\theta \|^2_{Z'} &= \sum_{k\geq 0} \| (\phi^{\star})^k \|^2 \| C^{\star}\theta \|^2_{Z'} \\ &\leq M \| C^{\star}\theta \|^2_{Z'} \text{ with } M > 0 \end{aligned}$$

using (16). Since $(S_{0,d})$ is exactly controllable asymptotically, there exists $\gamma > 0$ such that:

$$\| C^{\star}\theta \|^2_{Z'} \leq \gamma^2 \sum_{k\geq 0} \| B^{\star}(\phi^*)^k C^{\star}\theta \|^2_{\mathcal{U}'}$$

consequently, there exists $\eta = M\gamma^2$ such that:

$$\sum_{k\geq 0} \| (\phi^*)^k C^{\star}\theta \|^2_{Z'} \leq \eta \sum_{k\geq 0} \| B^{\star}(\phi^*)^k C^{\star}\theta \|^2_{\mathcal{U}''}$$

The result is then given by proposition 3.2. □

Remark 5.5 *The converse is not true.*

We also have the following analogous result.

Proposition 5.6 *If* $(S_{0,d})$ *is weakly controllable asymptotically, then* $(S_d) + (E_d)$ *is weakly remediable asymptotically.*

Proof:
$(S_d) + (E_d)$ weakly remediable asymptotically is equivalent to

$$ker[B^\star(R_C^\infty)^\star)] = ker[(R_C^\infty)^\star]$$

i.e.

$$ker[B^\star(R_C^\infty)^\star] \subset ker[(R_C^\infty)^\star]$$

or equivalently

$$ker[(H^\infty)^\star C^\star] \subset ker[(R_C^\infty)^\star]$$

because $(H^\infty)^\star C^\star = B^\star(R_C^\infty)^\star$. Let $\theta \in ker[(H^\infty)^\star C^\star]$, we have $[(H^\infty)^\star C^\star]\theta = 0$, then $(C^\star)\theta = 0$ because $ker[(H^\infty)^\star] = \{0\}$, then $\theta \in ker(C^\star)$.

Since $ker(C^\star) \subset ker[(R_C^\infty)^\star]$, we have the result. □

We introduce hereafter the notion of strategic actuators in the asymptotic case.

Definition 5.7 *Actuators are said to be strategic asymptotically if the corresponding system* $(S_{0,d})$ *is weakly controllable asymptotically.*

From proposition 5.6, we deduce that also in the asymptotic case, strategic actuators are efficient.

As it will be seen hereafter, the converse is not true. Let us note that in the case of a discrete version of a diffusion process, i.e. where the operator ϕ is defined by (32), the results are similar to those obtained in the finite horizon case, using the analyticity property. In this case:

i) Actuators $(\Omega_i, g_i)_{i=1,p}$ are strategic if and only if

$$rank M_n = r_n; \forall n \geq 1 \tag{49}$$

In the case where all the eigenvalues are simple, i.e.

$$r_n = 1 \ ; \ \forall n \geq 1 \tag{50}$$

then $(\Omega_i, g_i)_{i=1,p}$ are strategic asymptotically if and only if

$$rank(M_n) = 1; \forall n \geq 1$$

For $p = 1$, (Ω_1, g_1) is asymptotically strategic if and only if

$$\langle g_1, \varphi_{nj} \rangle \neq 0 \ ; \ \forall n \geq 1 \tag{51}$$

ii) The rank condition (49) is not necessary to have efficient actuators. For example, in a two dimension case where the domain is a disk [4,5,6,7], one need at least $p = 2$ to have strategic actuators, but one actuator can be efficient. Moreover, in the case where the domain is a square, any finite number of actuators can not be strategic, but here also, one actuator can be efficient.

iii) Actuators can be efficient without being strategic. Indeed, in the case of a one dimension system with $\Omega =]0, 1[$, then for example $g = \varphi_{n_0}$, the actuator (Ω, g) is not strategic, using (51). But if the sensor (D, h) is such that

$$\langle h, \varphi_{n_0} \rangle \neq 0$$

the actuator (Ω, g) is strategic using proposition 3.10.

In the general case, and for p zone actuators $(\Omega_i, g_i)_{1 \leq i \leq p}$, we have the following characterization result:

Proposition 5.8 *Actuators $(\Omega_i, g_i)_{\ 1 \leq i \leq p}$ are strategic asymptotically if and only if*

$$\left\langle \phi^k g_i, z \right\rangle = 0; \ \forall k \geq 0, \ \forall i = 1, ..., p \ \Rightarrow z = 0$$

Proof: Derives from the fact that

$$(H^\infty)^* = (B^*, B^*\phi^*, ..., B^*(\phi^*)^k........)^{tr}$$

5.2 Asymptotic remediability and stabilizability

In this section, we study the relationship between the notions of stabilizability and asymptotic remediability. This relation depend on the choice of the sensors and is not so clear than the previous one with the asymptotic controllability. Let us note that the problem of asymptotic remediability can be considered and solved for non stable systems, and as it will be shown, a non stable system may be asymptotically remediable without being stabilizable.

We recall hereafter the notion of stabilizability.

Definition 5.9
The system $(S_{0,d})$ given by (48) is said to be exponentially stabilizable if there exists a sequence of feedback controls

$$u_k = -Fz_k \quad \text{with } F \in \mathcal{L}(Z, U) \quad \text{and } k \geq 0$$

such that the system

$$\begin{cases} x_{k+1} & = & (\phi - BK)x_k \ ; \ \ k \geq 0 \\ x_0 & \in & Z \end{cases} \tag{52}$$

is exponentially stable.

Let us note that the considered system can be a discrete version of a continuous time one. Indeed, if we consider the system described by the following linear state equation

$$\begin{cases} \dot{z}(t) = Az(t) + \mathcal{B}u(t) + g(t) \ ; \ \ t > 0 \\ z(0) = z_0 \end{cases} \tag{53}$$

augmented by the following output equation:

$$y(t) = Cz(t) \ ; \ \ t \geq 0 \tag{54}$$

where A generates a strongly continuous semi-group (s.c.s.g) $(S(t))_{t\geq 0}$, $\mathcal{B} \in \mathcal{L}(U, Z)$, $C \in \mathcal{L}(Z, Y)$, $g \in L^2(0, +\infty; Z)$; $u \in L^2(0, +\infty; U)$. For $\tau > 0$ sufficiently small, the corresponding discrete version is as follows

$$\begin{cases} z_{k+1} = \phi z_k + Bu_k + f_k \\ z_0 \in Z \ ; \quad\ k \geq 0 \end{cases} \tag{55}$$

augmented by the output equation, also noted (E_d)

$$y_k = Cz_k \ ; \quad \text{k} \geq 0 \tag{56}$$

where $z_k = z(k\tau)$; $u_k = u(k\tau)$ and

$$\phi = S(\tau) \quad ; \quad Bu_k = \int_0^\tau S(\tau - s)\mathcal{B}u_k(s)ds \quad ; \quad f_k = \int_0^\tau S(\tau - s)g_k(s)ds \quad (57)$$

u_k and g_k are respectively the restrictions of u and g to the time interval $[k\tau, (k+1)\tau[$. If τ is small, u_k and g_k can be assumed to be constant on the interval $[k\tau, (k+1)\tau[$.

Concerning the stability, if ϕ is defined by (32), i.e. A given by (33), then the system

$$\begin{cases} z_{k+1} = \phi z_k \\ z_0 \in Z \quad ; \quad k \geq 0 \end{cases} \quad (58)$$

is exponentially stable if and only if

$$\sup_{n\geq 1} Re(\lambda_n) < 0 \quad (59)$$

For a non stable system, we have the following result [8,9]

Proposition 5.10

We assume that there exist a finite number $n_0 \geq 1$ of non negative eigenvalues noted $\lambda_1, ..., \lambda_{n_0}$. Then the system $(S_{0,d})$ excited by p actuators $(\Omega_i, g_i)_{i=1,p}$ is stabilizable if and only if

i) $p \geq \sup_{1\leq n\leq n_0} r_n$

ii) $rankM_n = r_n$ *for* $1 \leq n \leq n_0$, *with* M_n *defined in (38).*

The proof is similar to that established in the continuous case.

As it will be shown in the following paragraph, a system may be remediable asymptotically but not stabilizable.
But for a non convenient choice of sensors and actuators, the system may be stabilizable without being remediable asymptotically. Various other situations are also examined.

6 Case of a discrete version of a diffusion process

In this part, we examine the case of discrete version of a diffusion system respectively with a Dirichlet and a Neumann boundary conditions [5,6, ...,11]. We consider without loss of generality a one dimension, the results are similar in a higher space dimension (for example if Ω is rectangle or a disk).

6.1 Dirichlet case

We consider the system excited by p zone actuators

$$(S_{1,c})\begin{cases} \dfrac{\partial z(x,t)}{\partial t} &= \Delta z(x,t) + \displaystyle\sum_{i=1}^{p} g_i(x)u_i(t) + G(x,t) \text{ in } \Omega\times]0,+\infty[\\ z(x,0) &= z_0(x) \text{ in } \Omega \\ z(x,t) &= 0 \text{ on } \partial\Omega\times]0,+\infty[\end{cases}$$

with $G \in L^2(0,+\infty;Z)$; $u \in L^2(0,+\infty;U)$.

The discrete version of (6.1) is as follows

$$(S_{1,d})\begin{cases} z_{k+1} = \phi z_k + Bu_k + f_k \\ z_0 \in Z \quad ; \qquad k \geq 0 \end{cases} \tag{60}$$

It is augmented by the discrete output equation given by q zone sensors

$$(E_{1,d}) \;\; y = (\langle h_1, z\rangle, \cdots, \langle h_q, z\rangle)^{tr}$$

For $\Omega =]0,1[$, the Laplacian operator Δ admits an orthonormal basis of eigenfunctions defined by

$$\varphi_n(\xi) = \sqrt{2}\sin(n\pi\xi); n \geq 1$$

The associated eigenvalues are simple ($r_n = 1$ for $n \geq 1$) and given by

$$\lambda_n = -n^2\pi^2; n \geq 1$$

Δ generates a s.c.s.g. $(S(t))_{t\geq 0}$ defined by:

$$S(t)z = \sum_{n=1}^{+\infty} e^{-n^2\pi^2 t}\langle z, \varphi_n\rangle\varphi_n \tag{61}$$

which is self-adjoint and exponentially stable. In this case, the operator ϕ is given by

$$\phi z = \sum_{n=1}^{+\infty} e^{-n^2\pi^2\tau}\langle z, \varphi_n\rangle\varphi_n \tag{62}$$

and the system (60), with $u = 0$ and $f = 0$, is **exponentially stable**. The operators H^{∞} and $\overline{H^{\infty}}$ defined respectively on $\ell^2(U)$ and $\ell^2(Z)$ by

$$H^{\infty}u = \sum_{k=0}^{+\infty} \phi^k B u_k \quad and \quad \overline{H^{\infty}} f = \sum_{k=0}^{+\infty} \phi^k f_k \tag{63}$$

are well defined. The asymptotic controllability is then well defined. On the other hand

$(S_{1,d}) + (E_{1,d})$ is exactly remediable asymptotically if and only if, there exists $\gamma > 0$ such that: for all $\theta = (\theta_1, \cdots, \theta_q) \in I\!R^q$, we have using corollary 3.6

$$\sum_{n\geq 1} \frac{1}{2n^2\pi^2} (\sum_{i=1}^{q} \theta_i \langle h_i, \varphi_n \rangle)^2 \leq \gamma \sum_{j=1}^{p} \sum_{k\geq 1} (\sum_{n\geq 1} e^{-n^2\pi^2 k\tau} \langle g_j, \varphi_n \rangle \sum_{i=1}^{q} \theta_i \langle h_i, \varphi_n \rangle)^2$$

In the case of one sensor and one actuator ($p = q = 1$), this inequality becomes

$$\sum_{n\geq 1} \frac{1}{2n^2\pi^2} [\theta \langle h, \varphi_n \rangle]^2 \leq \gamma \sum_{k\geq 1} [\sum_{n\geq 1} e^{-n^2\pi^2 k\tau} \langle g, \varphi_n \rangle \theta \langle h, \varphi_n \rangle]^2$$

or

$$\sum_{n\geq 1} \frac{1}{2n^2\pi^2} [\langle h, \varphi_n \rangle]^2 \leq \gamma \sum_{k\geq 1} [\sum_{n\geq 1} e^{-n^2\pi^2 k\tau} \langle g, \varphi_n \rangle \langle h, \varphi_n \rangle]^2$$

$(S_{1,d}) + (E_{1,d})$ is exactly remediable asymptotically for example if $g = h = \varphi_{n_0}$ with $n_0 \geq 1$. But the corresponding system $(S_{1,d})$ is not exactly controllable asymptotically because it is not weakly controllable asymptotically.

Now, in the case of one actuator (Ω_1, g_1) and one sensor (D, h), consider $n_0 \geq 1$ such that $\langle h, \varphi_{n_0} \rangle \neq 0$, we have $rank(G^{tr}_{n_0}) = 1$.
(Ω_1, g_1) is then efficient if $\langle g_1, \varphi_{n_0} \rangle \neq 0$, or equivalently

$$\int_{\Omega_1} g_1(\xi) sin(n\pi\xi) d\xi \neq 0$$

For example, if $g_1 = \varphi_{n_0}$, (Ω_1, g_1) is efficient (using (42)), but it is not strategic because the condition

$$\int_0^1 g_1(\xi) sin(n\pi\xi) d\xi \neq 0 \; ; \; n \geq 1$$

is not satisfied.

Remark 6.1 *In the Dirichlet case, the s.c.s.g. $(S(t))_{t\geq 0}$ is exponentially stable, the problem of stabilizability is not posed, but it will be considered in the Neumann case where $(S(t))_{t\geq 0}$ is not stable.*

6.2 Neumann case

In this par, we consider the following one dimension system with a Neumann boundary condition

$$(S_{2,c})\begin{cases} \dfrac{\partial z(x,t)}{\partial t} &= \Delta z(x,t) + G(x,t) + \displaystyle\sum_{i=1}^{p} g_i(x)u_i(t) \text{ in }]0,1[\times]0,+\infty[\\ z(x,o) &= z_0(x) \text{ in }]0,1[\\ \dfrac{\partial z(0,t)}{\partial x} &= \dfrac{\partial z(1,t)}{\partial x} = 0 \text{ in }]0,+\infty[\end{cases}$$

In this case, we have

$$S(t)z = \sum_{n\geq 0} e^{-n^2\pi^2 t} \langle z, \varphi_n\rangle \varphi_n$$

with

$$\varphi_n(\xi) = \sqrt{2}\cos(n\pi\xi); n \geq 1 \text{ and } \varphi_0 \equiv 1$$

The eigenvalues are given by

$$\lambda_n = -n^2\pi^2 \ ; \ n \geq 1 \text{ and } \lambda_0 = 0$$

$(S(t))_{t\geq 0}$ is not exponentially stable. The operator ϕ is given by

$$\phi z = \sum_{n=0}^{+\infty} e^{-n^2\pi^2\tau}\langle z, \varphi_n\rangle\varphi_n \tag{64}$$

In this case, the corresponding discrete system

$$(S_{2,d})\begin{cases} z_{k+1} = \phi z_k + Bu_k + f_k \\ z_0 \in Z \quad ; \qquad k \geq 0 \end{cases} \tag{65}$$

with $u = 0$ and $f = 0$, is not **exponentially stable** and the number of non negative eigenvalues is $n_0 = 1$. The operators H^∞ and $\overline{H^\infty}$ defined by

$$H^{\infty}u \;=\; \sum_{k=0}^{+\infty}\phi^k B u_k \quad and \quad \overline{H^{\infty}}f \;=\; \sum_{k=0}^{+\infty}\phi^k f_k \tag{66}$$

are not generally well defined respectively on $\ell^2(U)$ and $\ell^2(Z)$. Consequently, the asymptotic controllability is not well defined.

The system $(S_{2,d})$ is augmented by the output equation:

$$(E_{2,d})\;\; y_k = (\langle h_1, z_k\rangle, \cdots, \langle h_q, z_k\rangle)^{tr} \;\;;\quad k \geq 0$$

with $h_1, \cdots, h_q$ orthogonal to φ_0 , i.e. to the unstable part:

$$\langle h_i, \varphi_0\rangle = 0;\;\; 1 \leq i \leq q$$

The operators H_C^{∞} and R_C^{∞} are then well defined and the characterization results are similar to those obtained in the Dirichlet case.

Concerning the stabilizability, in the case of one actuator and using proposition 5.10, the system is stabilizable if and only if

$$rank M_0 = r_0 = 1$$

this is equivalent to

$$\langle g, \varphi_0\rangle \neq 0$$

As seen, the stability condition is not necessary for considering the asymptotic remediability. Moreover a system may be remediable asymptotically without being stabilizable. Indeed, for $g = \varphi_{n_0}$ with $n_0 \geq 1$, we have $\langle g, \varphi_0\rangle = 0$ and then the system is not stabilizable. But if also $h = \varphi_{n_0}$, we have

$$\langle h, \varphi_0\rangle = 0$$

The problem of asymptotic compensation is well posed. We have

$$\langle g, \varphi_{n_0}\rangle\langle h, \varphi_{n_0}\rangle = 1 \neq 0$$

The system is then remediable asymptotically.

Conversely, the system may be stabilizable but not remediable asymptotically. Hence, if $g = \varphi_0$, we have $\langle g, \varphi_0\rangle \neq 0$ and then the system is stabilizable. But for $h = \varphi_{n_0}$ with $n_0 \geq 1$, we have

$$\langle g, \varphi_n\rangle\langle h, \varphi_n\rangle = 0 \;;\; \forall n \geq 1$$

and then the system is not asymptotically remediable.

Let us note that with a convenient choice of actuators and sensors, a non stable system may be stabilizable and remediable asymptotically. But it may be also non stabilizable and non remediable asymptotically. These situations are illustrated here after:

- For $g = \varphi_1$, the system is not stabilizable because $\langle g, \varphi_0 \rangle = 0$. On the other hand, if $h = \varphi_2$, we have

$$\langle g, \varphi_n \rangle \langle h, \varphi_n \rangle = 0 \ ; \ \forall n \geq 1$$

then, the system is not also asymptotically remediable.

- For $g(x) = 2x$, we have $\langle g, \varphi_0 \rangle = 1$. consequently, the system is stabilizable. But if $h = \varphi_1$, we have

$$\langle h, \varphi_0 \rangle = 0$$

The problem of asymptotic compensation is well posed. We have

$$\langle g, \varphi_1 \rangle \langle h, \varphi_1 \rangle \neq 0$$

and hence, the system is also asymptotically remediable.

7 Approximations and numerical simulations

This section is consecrated to numerical approximations and simulations of the asymptotic compensation problem. We give an approximation of θ_f as a solution of a finite dimension linear system $Ax = b$, and then the optimal control u_{θ_f}, with a comparison between the corresponding observation noted $y_{(u_{\theta_f}, f)}$, the normal one (i.e. $y_{(0,0)}$) and $y_{(0,f)}$).

7.1 Approximations

• **Coefficients of the system :**

For $i, j \geq 1$, let

$$a_{ij} = < \sum_{I=0}^{+\infty} Q(I) Q^*(I) e_i, e_j >_{R^q}$$

where $Q(I)$ is the operator

$$\begin{array}{rccc} Q(I): & R^p & \longrightarrow & R^q \\ & x & \longrightarrow & C\phi^k Bx \end{array}$$

and $(e_i)_{1\leq i\leq q}$ is the canonical basis of R^q. We have

$$Q(I)x = C\Phi^I Bx = C\sum_{m\geq 1} e^{I\lambda_m \tau} \sum_{l=1}^{r_m} < Bx, \varphi_{ml} >_{L^2(\Omega)} \varphi_{ml}$$

and

$$\begin{array}{rccc} Q^*(I): & R^q & \longrightarrow & R^p \\ & y & \longrightarrow & \displaystyle\sum_{m\geq 1} e^{I\lambda_m \tau} \sum_{l=1}^{r_m} < \varphi_{ml}, C^* y >_{L^2(\Omega)} B^* \varphi_{ml} \end{array}$$

then

$$Q(I)Q^*(I)e_i =$$

$$C\sum_{m\geq 1}\sum_{l=1}^{r_m}\sum_{n\geq 1}\sum_{k=1}^{r_n} e^{I\lambda_m \tau} e^{I\lambda_n \tau} < BB^* \varphi_{nk}, \varphi_{ml} >_{L^2(\Omega)} < \varphi_{nk}, C^* e_i >_{L^2(\Omega)} \varphi_{ml}$$

and

$$\begin{aligned} a_{ij} &= \sum_{m\geq 1}\sum_{l=1}^{r_n}\sum_{n\geq 1}\sum_{k=1}^{r_n}\sum_{s=1}^{p} \frac{1}{1-e^{(\lambda_n+\lambda_m)\tau}} < g_s, \varphi_{nk} >_{L^2(\Omega_s)} < g_s, \varphi_{ml} >_{L^2(\Omega)} . \\ &\qquad\qquad < \varphi_{nk}, h_i >_{L^2(\Omega)} < \varphi_{ml}, h_j >_{L^2(D_j)} \\ &\simeq \sum_{m=1}^{M}\sum_{l=1}^{r_n}\sum_{n=1}^{N}\sum_{k=1}^{r_n}\sum_{s=1}^{p} \frac{1}{1-e^{(\lambda_n+\lambda_m)\tau}} < g_s, \varphi_{nk} >_{L^2(\Omega_s)} < g_s, \varphi_{ml} >_{L^2(\Omega)} . \\ &\qquad\qquad < \varphi_{nk}, h_i >_{L^2(\Omega)} < \varphi_{ml}, h_j >_{L^2(D_j)} \end{aligned}$$

for M, N sufficiently large, and

$$\begin{aligned} b_j &= - < R_C^\infty f, e_j >_{\mathbb{R}^q} \\ &= -\sum_{n\geq 1}\sum_{k=1}^{r_n}\sum_{I=0}^{\infty} \left(e^{I\lambda_n \tau} < f_I(.), \varphi_{nk} >_{L^2(\Omega)} < h_j, \varphi_{nk} >_{L^2(D_j)} \right) \\ &\simeq -\sum_{n\geq 1}^{N}\sum_{k=1}^{r_n}\sum_{I=0}^{\infty} \left(e^{I\lambda_n \tau} < f_I(.), \varphi_{nk} >_{L^2(\Omega)} < h_j, \varphi_{nk} >_{L^2(D_j)} \right) \end{aligned}$$

• **Optimal control :**
In this part, we give an approximation of the optimal control u_{θ_f}, we have

$$u_{I,\theta_f} = B^*(\Phi^*)^I C^*\theta_f$$

The function coordinates of $u_{j,\theta_f}(.)$ are given by

$$\begin{aligned} u^j_{I,\theta_f} &= \sum_{n\geq 1}\sum_{k=1}^{r_n}\sum_{i=1}^{q} e^{I\lambda_n\tau}\theta_{i,f} < h_i,\varphi_{nk} >_{L^2(\Omega)}>< g_j,\varphi_{nk} >_{L^2(\Omega_j)} \\ &\simeq \sum_{n\geq 1}^{N}\sum_{k=1}^{r_n}\sum_{i=1}^{q} e^{I\lambda_n\tau}\theta_{i,f} < h_i,\varphi_{nk} >_{L^2(\Omega)}>< g_j,\varphi_{nk} >_{L^2(\Omega_j)} \end{aligned}$$

for a large integer N.

• **Cost:** The minimum energy (cost) is defined by $\| u_{\theta_f} \|_{l^2(U)}$

$$\| u_{\theta_f} \|_{l^2(U)} = \left(\sum_{I=0}^{\infty} \|B^*(\Phi^*)^I C^*\theta_f\|^2_{\mathbb{R}^p}\right)^{\frac{1}{2}}$$

$$= \left(\sum_{j=1}^{p}\left(\sum_{n\geq 1}\sum_{k=1}^{r_n}\sum_{i=1}^{q}\frac{1}{1-e^{\lambda_n\tau}}\theta_{i,f} < h_i,\varphi_{nk} >_{L^2(\Omega)}< g_j,\varphi_{nk} >_{L^2(\Omega_j)}\right)^2\right)^{\frac{1}{2}}$$

$$\simeq \left(\sum_{j=1}^{p}\left(\sum_{n\geq 1}^{N}\sum_{k=1}^{r_n}\sum_{i=1}^{q}\frac{1}{1-e^{\lambda_n\tau}}\theta_{i,f} < h_i,\varphi_{nk} >_{L^2(\Omega)}< g_j,\varphi_{nk} >_{L^2(\Omega_j)}\right)^2\right)^{\frac{1}{2}}$$

for N sufficiently large.

• **The corresponding observation :**

The corresponding observation is given by

$$y_{I,u_{\theta_f}} = Cz_{I,u_{\theta_f}} = C\left(\Phi^I z_0 + \sum_{J=0}^{I}\Phi^J B u_{J,\theta_f} + \sum_{J=0}^{I}\Phi^J B f_J(.)\right)$$

Its coordinates $\left(\left(y^j_{I,u_{\theta_f}}\right)_{I\geq 0}\right)_{j=1,q}$ are obtained as follows:

$$\begin{aligned} y^j_{I,u_{\theta_f}} &= \sum_{n\geq 1}\sum_{k=1}^{r_n} e^{I\lambda_n\tau} < z_0, \varphi_{nk} >_{L^2(\Omega)} < h_j, \varphi_{nk} >_{L^2(D_j)} \\ &\quad + \sum_{n\geq 1}\sum_{k=1}^{r_n}\sum_{i=1}^{p} < g_i, \varphi_{nk} >_{L^2(\Omega)} < h_j, \varphi_{nk} >_{L^2(D_j)} \sum_{J=0}^{I} e^{J\lambda_n\tau} u^i_{J,\theta_f} \\ &\quad + \sum_{n\geq 1}\sum_{k=1}^{r_n} < h_j, \varphi_{nk} >_{L^2(D_j)} \sum_{J=0}^{I} e^{J\lambda_n\tau} < f_J(.), \varphi_{nk} >_{L^2(\Omega)} \\ &\simeq \sum_{n=1}^{N}\sum_{k=1}^{r_n} e^{I\lambda_n\tau} < z_0, \varphi_{nk} >_{L^2(\Omega)} < h_j, \varphi_{nk} >_{L^2(D_j)} \\ &\quad + \sum_{n=1}^{N}\sum_{k=1}^{r_n}\sum_{i=1}^{p} < g_i, \varphi_{nk} >_{L^2(\Omega)} < h_j, \varphi_{nk} >_{L^2(D_j)} \sum_{J=0}^{I} e^{J\lambda_n\tau} u^i_{J,\theta_f} \\ &\quad + \sum_{n=1}^{N}\sum_{j=1}^{r_n} < h_j, \varphi_{nk} >_{L^2(D_j)} \sum_{J=0}^{I} e^{J\lambda_n\tau} < f_J(.), \varphi_{nk} >_{L^2(\Omega)} \end{aligned}$$

for N sufficiently large.

7.2 Numerical simulations

7.2.1 Dirichlet case

We consider the discrete version $(S_{1,d})$ of system $(S_{1,c})$ in the one dimension case with $\Omega =]0,1[$ and a Dirichlet boundary condition. In this case, the functions $\varphi_n(.)$ are defined by

$$\varphi_n(\xi) = \sqrt{2} sin(n\pi\xi)\ ;\ \ n \geq 1$$

The associated eigenvalues are simple and given by

$$\lambda_n = -n^2\pi^2\ ;\ \ n \geq 1$$

Then for

⋆ an initial state: $z_0(.) \equiv 0$,

⋆ a sensor (D,h) with $D =]0,1[$ and $h(\xi) = \sqrt{2}\xi^2$ $(q = 1)$

⋆ an efficient actuator (Ω, g) with $\Omega =]0,1[$ and $g(\xi) = 2\xi^3$

⋆ a disturbance defined as follows

$$f(I,\xi) = \begin{cases} \sin(I)e^{\frac{n}{10}I\xi} \text{ if } I = 0,1,\ldots 100 \\ \frac{\sin(I+1)}{(I+1)^2}e^{-\frac{1}{10}I\xi} \text{ if } I > 100 \end{cases}$$

⋆ $M = N = 10$

⋆ $\tau = 0,02$

we obtain numerical results which illustrate those established previously.

To simplify the notations, let us note $y(u,f) \equiv (y_I(u,f))_{I\geq 0}$ the discrete observation corresponding to the control u and the disturbance f. Hence $y(0,f)$ (respectively $y(0,0)$) will represent the observation for $u = 0$ (respectively the normal observation, i.e. $u = 0$ and $f = 0$).

The figure 1 represents the evolution of the observations $y(u,f)$, $y_{(0,f)}$ and $y_{(0,0)} \equiv 0$ with respect to I. This figure show that for I sufficiently large ($I \geq 25$), we have

$$y_I(u_{\theta_f}, f) \simeq y_I(0,0)$$

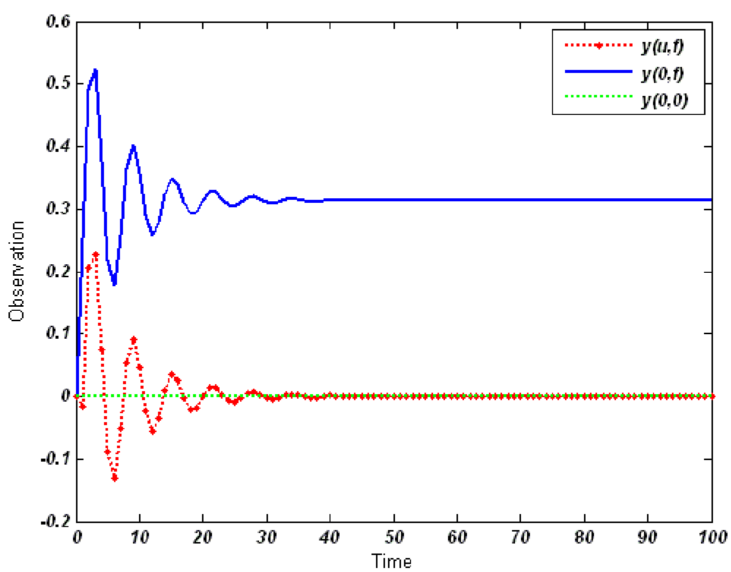

Figure 1: Representation of $y_{(u_{\theta_f},f)}$, $y_{(0,0)}$ and $y_{(0,f)}$ in the Dirichlet case.

The optimal control u_{θ_f} ensuring the asymptotic compensation of the disturbance is represented, with respect to I, in figure 2.

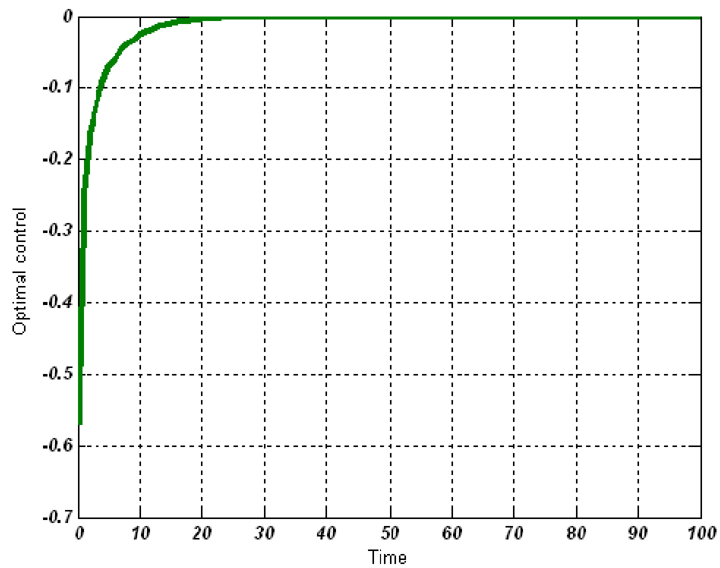

Figure 2: The optimal control: Dirichlet case.

7.2.2 Neuman case

We consider the system $(S_{2,d})$ with $\Omega =]0,1[$ and a Neuman boundary condition, in this case

$$\varphi_n(\xi) = \sqrt{2}cos(n\pi\xi)\ ;\quad n \geq 1 \ \text{ and } \varphi_0 \equiv 1$$

and

$$\lambda_n = -n^2\pi^2\ ;\quad n \geq 1 \ \text{ and } \lambda_0 = 0$$

For

$\star$ an initial state: $z_0(.) \equiv 0$,

$\star$ a sensor: (D,h), with $D =]0,1[$ and $h(\xi) = \xi e^{\xi}$

$\star$ an efficient actuator (Ω, g) with $\Omega =]0,1[$ and $g(\xi) = cos(\frac{\Pi}{3}\xi)$

$\star$ a disturbance function defined by

$$f(I,\xi) = 480\frac{cos(I+1)\frac{\Pi}{2}}{(I+1)^2}e^{-\frac{1}{10}I\xi} \quad \text{for } I \geq 0$$

$\star$ $M = N = 10$

$\star$ $\tau = 10^{-3}$

we obtain similar numerical results. In figure 3, the observations $y_{(u_{\theta_f},f)}$, $y_{(0,0)}$ and $y_{(0,f)}$ are represented. We remark that also in the Neumann case, we have

$$y_I(u_{\theta_f}, f) \simeq y_I(0,0) \quad \text{for } I \geq 40$$

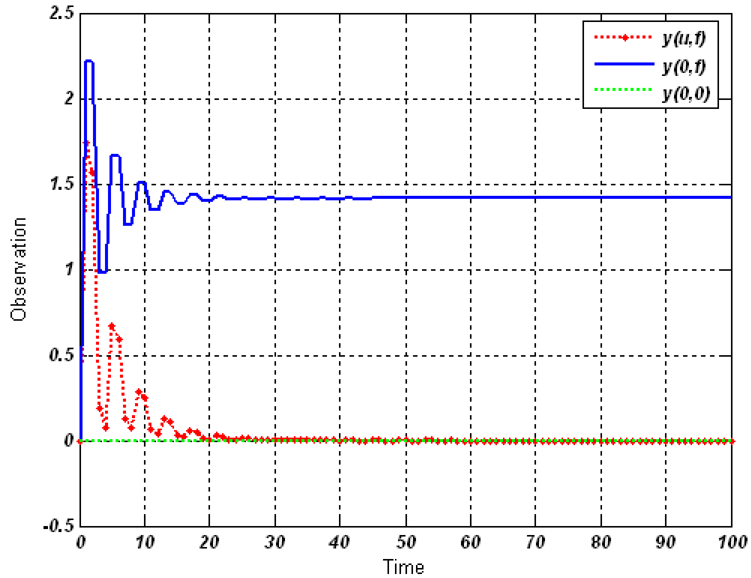

Figure 3: Representation of $y(u_{\theta_f}, f))$, $y(0,0)$ *and* $y(0,f)$ *in the Neuman case.*

The figure 4 represents the optimal control u_{θ_f}.

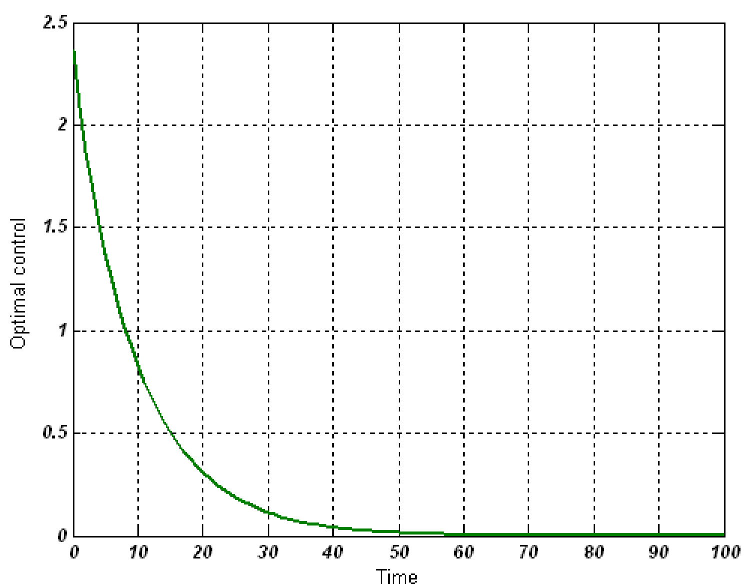

Figure 4: The optimal control: Neuman case.

Conclusion

In this papers which is an extension of previous works to the asymptotic discrete case, we have firstly introduced and characterized the notions of asymptotic weak and exact remediability as well as asymptotic efficient actuators. Then, using an extension of the Hilbert Uniqueness Method, we have shown how to find the optimal control ensuring the asymptotic compensation of a known or unknown disturbance, and given a characterization of the disturbances which are exactly remediable asymptotically.

We have also defined and characterized the controllability and strategic actuators in the discrete asymptotic case.

We have shown that also in the discrete asymptotic case, the controllability remain stronger than the remediability, and hence that asymptotic strategic actuators are asymptotically efficient. The converse is not true.

The relationship between the asymptotic remediability and the stabilizability is also studied. This relation depend on the choice of the actuators and the sensors. The discrete version of diffusion system is examined. Illustrative examples, numerical approximations and results are also presented.

The results are developed for a class of discrete linear distributed systems and for zone actuators and sensors, but the considered approach can be extended to unbounded operators (pointwise sensors and actuators) with a convenient choice of spaces, and also to other classes of systems or other similar problems.

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