Applied Mathematical Sciences, Vol. 2, 2008, no. 30, 1479 - 1486

Dynamic DEA and its Development for Additive Model to Evaluate Return to Scale

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Abstract

Quasi-fixed inputs are incorporated into dynamic DEA, A unique feature of the quasi-fixed inputs is that those are considered as outputs at the current, while being treated as inputs at the next period. In this paper we propose dynamic Additive model when a firm employs quasi-fixed inputs, then obtain return to scale in dynamic Additive model.

Keywords: Dynamic DEA, return to scale

1. Introduction

Data envelopment analysis (DEA), first proposed by charnes et al. [2], is a new technique developed in operations research and management science over the last two decades for measuring productive efficiency. This is a nonparametric technique based only on the observed input-output data of firms. The introduction of quasi-fixed inputs into a DEA model can be seen as a first step toward dynamic DEA and Malmquist indices. Nemoto and Goto [2] extended DEA to a dynamic framework. They incorporates two different type of inputs (variable input and quasi-fixed inputs) into dynamic DEA. Sueyoshi and Sekitani [5] extended the dynamic DEA of Nemoto and Goto [3] in a manner that the concept of return to scale is incorporated into the dynamic DEA. Based on the work of Sueyoshi and Sekitani we obtain return to scale in dynamic

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Additive model. The remainder of this paper is divided in to section: Section 2 begin with dynamic DEA. Then in section 3, dynamic Additive model is introduced. The next section return to scale in dynamic Additive model is considered. Section 5 applies an example in volving 11 Iranian gas companies in two period. Conclusions appear in Section 6.

2. Dynamic DEA

Let x_t denote a $m\times 1$ vector of variable inputs used in the period t, k_{t-1} a $l\times 1$ vector of quasi-fixed inputs at the beginning of the period t, y_t a $s\times 1$ vector of outputs produced in the period t, and k_t a $l\times 1$ vector of quasi-fixed inputs at the end of the period t. In the dynamic DEA, OMU$_p$ is characterized as a production process from $(x_{tp}, k_{t-1p}) \in R^{m+l}$ to $(y_{tp}, k_{tp}) \in R^{s+l}$. Furthermore a production possibility set in the period t specified as follows:

$$\varphi_t = \{(x_t, k_{t-1}, y_t, k_t) \in R^{m+2l+s} | (x_t, k_{t-1}) \text{ can yield } (y_t, k_t)\}.$$

It is required that φ_t satisfies the regularity conditions:
(i) if $(\bar{x}_t, \bar{k}_{t-1}, y_t, k_t) \in \varphi_t$ and $(\bar{x}_t, \bar{k}_t) \le (x_t, k_{t-1})$, the $(x_t, k_{t-1}, y_t, k_t) \in \varphi_t$,
(ii) if $(x_t, k_{t-1}, \bar{y}_t, \bar{k}_t) \in \varphi_t$ and $(\bar{y}_t, \bar{k}_t) \ge (y_t, k_t)$, then $(x_t, k_{t-1}, y_t, k_t) \in \phi_t$,
(iii) φ_t is closed and convex.

Suppose that in period t, there are N observations regarding inputs and outputs: variable input, $X_t = (x_{t_1}, \ldots, x_{tN})$, quasi-fixed onputs at the beginning of the period t, $K_{t-1} = (k_{t-11}, \ldots, k_{t-1N})$, Variable output, $Y_t = (y_{t1}, \ldots, y_{tN})$, and quasi-fixed inputs at the end of the period t, $K_t = (k_{t1}, \ldots, k_{tN})$. A vector of weights, $\lambda_t = (\lambda_{t1}, \ldots, \lambda_{tN})$ is used to connect input and output of N DMU$_s$. It is known that the smallest set including N observations and satisfing (i)-(iii) take the form:

$$\varphi_t =$$

$$\{(x_t, k_{t-1}, y_t, k_t) | X_t\lambda_t \le x_t, K_{t-1}\lambda_t \le k_{t-1}, K_t\lambda_t \ge k_t, y_t\lambda_t \ge y_t, e\lambda_t = 1, \lambda_t \ge 0\}.$$

An important feature of DEA dynamics is that its objective is formulated to minimize a total production cost over an entire observed Period. Nemotu and Goto [3] use the following formulation to determine the cost minimum of the pth DMU:

$$
\begin{aligned}
\text{Min} \quad & \sum_{t=1}^{T} \gamma_t (w_t x_t + v_t k_{t-1}) & & \\
s.t. \quad & X_t \lambda_t \le x_t, & & t = 1, \dots, T, \\
& K_{t-1} \lambda_t \le k_{t-1},, & & t = 1, \dots, T, \\
& K_t \lambda_t - \gamma_t \ge k_t, & & t = 1, \dots, T-1, \qquad (1) \\
& Y_t \lambda_t \ge y_{tp}, & & t = 1, \dots, T, \\
& e\lambda_t = 1, & & t = 1, \dots, T, \\
& k_0 = \bar{k}_0, k_t \ge 0, k_t \ge 0, \lambda_t \ge 0, & & t = 1, \dots, T_0
\end{aligned}
$$

where γ_t is a constant discount factor, w_t and v_t are prices of the variable and quasi-fixed inputs at the period t, respectively, Here the initial values of quasi-fixed input k_0 are given at $\bar{k}_0$.

3. Dynamic Additive Model

We combine input-oriented and output-oriented dynamic model in a dynamic single model, called the dynamic Additive model. Let there are n decision making units and their production activities are examind in T periods in the tth period, each DMU$_p$ uses two different type of inputs: $x_{t,p}$ and $k_{t-1,p}$ to yield two different types of outputs: $y_{t,p}$ and $k_{t,p}$. The inputs $k_{t-1,p}$ comes from the $(t-1)$th period and $k_{t,p}$ is used as input to the later period $t+1$.

Now define dynamic Additive model as:

$$
\begin{aligned}
S = \text{Max} \quad & \sum_{t=1}^{T} w_t s_t + \sum_{t=1}^{T} v_t \sigma_t + \sum_{t=1}^{T} u_t \gamma_t + \sum_{t=1}^{T} g_t z_t & & \\
s.t. \quad & X_t \lambda_t + s_t = x_{tp}, & & t = 1, \dots, T, \\
& K_{t-1} \lambda_t + \sigma_t = k_{t-1p}, & & t = 1, \dots, T, \qquad (2) \\
& K_t \lambda_t - \gamma_t = k_{tp}, & & t = 1, \dots, T-1, \\
& Y_t \lambda_t - z_t = y_{tp}, & & t = 1, \dots, T, \\
& e\lambda_t = 1, & & t = 1, \dots, T, \\
& \lambda_t \ge 0, s_t \ge 0, z_t \ge 0, \sigma_t \ge 0, & & t = 1, \dots, T, \\
& \gamma_t \ge 0, & & t = 1, \dots, T-1.
\end{aligned}
$$

The following notation are used in this model:
s_t: vector of slack variable corresponding to inputs $x_{t,p}$,

σ_t: vector of slack variable corresponding to inputs $k_{t-1,p}$,
γ_t: vector of slack variable corresponding to outputs $k_{t,p}$,
z_t: vector of slack variable corresponding to outputs $y_{t,p}$,
w_t: the slack variable weight vector of tth period corresponding to s_t,
v_t: the slack variable weight vector of tth period corresponding to σ_t,
u_t: the slack variable weight vector of tth period corresponding to γ_t,
g_t: the slack variable weight vector of tth period corresponding to z_t,
the other notations corresponding to [3].

In model (2) the objective is maximize the weighted sum of slack variable corresponding to DMU_p in the whole periods.

DMU_p in the whole periods is pareto efficient if and only if $S = 0$.

Defind the weights as follows:

$$w_{ti} = \frac{1}{\max\limits_{1\le j\le n}\{x_{tij}\}}\ \begin{matrix} t=1,\dots,T \\ i=1,\dots,m \end{matrix},\ v_{th} = \frac{1}{\max\limits_{1\le j\le n}\{k_{t-1hj}\}}\ \begin{matrix} t=1,\dots,T \\ h=1,\dots,l \end{matrix}$$

$$u_{th} = \frac{1}{\max\limits_{1\le j\le n}\{k_{thj}\}}\ \begin{matrix} t=1,\dots,T \\ h=1,\dots,l \end{matrix},\ g_{tr} = \frac{1}{\max\limits_{1\le j\le n}\{y_{trj}\}}\ \begin{matrix} t=1,\dots,T \\ r=1,\dots,s \end{matrix}$$

The dual model (2) is as follow:

$$\begin{array}{lll}
\text{Min} & \sum\limits_{t=1}^{T}\alpha_t x_{tp} + \sum\limits_{t=1}^{T}\beta_t k_{t-1} - \sum\limits_{t=1}^{T}\theta_t k_{tp} - \sum\limits_{t=1}^{T}\mu_t y_{tp} + \sum\limits_{t=1}^{T}\delta_t & \\
s.t. & \alpha_t X_t + \beta_t K_{t-1} - \theta_t K_t - \mu_t Y_t + e\delta_t \ge 0, & t=1,\dots,T, \\
& \alpha_T X_T + \beta_T K_{T-1} - \mu_T Y_T + e\delta_T \ge 0, & \\
& \alpha_t \ge w_t,\ \beta_t \ge v_t,\ \mu_t \ge g_t, & t=1,\dots,T, \\
& \theta_t \ge u_t, & t=1,\dots,T-1, \\
& \delta_t \text{free}, & t=1,\dots,T.
\end{array} \tag{3}$$

In this program the dual variables α_t are related to the first group, β_t are related to the second group, θ_t are related to the third group, μ_t are related to the fourth group and δ_t are related to the fifth group of constraints of (2).

4. Return to Scale in Dynamic Additive Model

Let DMU_p is pareto efficient otherwise use projection DMU_p. Now employ the sign of δ_t^* to portray the situation for return to scale. The production possibility set in tth period is defined as:

$$\varphi_t = \{(x_t, k_{t-1}, y_t, k_t) | X_t\lambda_t \le x_t, K_{t-1}\lambda_t \le k_{t-1}, Y_t\lambda_t \ge y_t, K_t\lambda_t \ge k_t, e\lambda_t = 1\}$$

A support hyperplane of φ_t at the point $(x_{tp}, k_{t-1p}, y_{tp}, k_{tp})$ is as:

$$H_t = \{(x_t, k_{t-1}, y_t, k_t) | \alpha_t^* x_t + \beta_t^* k_{t-1} - \theta_t^* k_t - \mu_t^* y_t + e\delta_t^* = 0\}.$$

First solve model (3) for determine return to scale of DMU_p in a fixed period, if $\delta_t^* = 0$ return to scale is constant if $\delta_t^* < 0$ then solve model follows:

$$\begin{aligned}
\text{Max} \quad & \hat{\delta}_t \\
s.t. \quad & \alpha_t X_t + \beta_t K_{t-1} - \theta_t K_t - \mu_t Y_t + e\hat{\delta}_t \geq 0, && t = 1, \ldots, T-1, \\
& \alpha_t X_t + \beta_t K_{t-1} - \mu_T Y_T + e\hat{\delta}_T \geq 0, && \\
& \alpha_t x_{tp} + \beta_t k_{t-1p} - \theta_t k_{tp} - \mu_t y_{tp} + e\hat{\delta}_t = 0, && t = 1, \ldots, T-1, \\
& \alpha_T X_{tp} + \beta_T k_{t-1p} - \mu_T y_{Tp} + e\hat{\delta}_T = 0, && \\
& \alpha_t \geq w_t,\ \beta_t \geq v_t,\ \mu_t \geq g_t,\ \hat{\delta}_t \leq 0, && t = 1, \ldots, T, \\
& \theta_t \geq u_t, && t = 1, \ldots, T-1.
\end{aligned} \tag{4}$$

The RTS on the tth period is as follows:
if $\hat{\delta}_t^* = 0$ return to scale is constant.
if $\hat{\delta}_t^* < 0$ return to scale is increasing.
if $\delta_t^* > 0$ had occurred of model (3) then the inequality $\hat{\delta}_t \leq 0$ in the model (4) would be replaced by $\hat{\delta} \geq 0$ and the objective in (4) would be reoriented to min $\hat{\delta}_t$ and return to scale determine from the following conditions,
if $\hat{\delta}_t^* = 0$ return to scale is constant.
if $\hat{\delta}_t^* > 0$ return to scale is devreasing.

5. A Simple Example

We apply the method to a data set consisting 11 gas companies located in 11 regions in Iran, [1]. The data for this analysis are derived from operations during 2003 and 2004. We use seven variable from the data set as inputs and outputs. Inputs include number of staff and budget, and outputs include amount of piping, number of new customer and amount of branch-line. Another type of output that is used as input in later period, is revenue. Each gas company uses the revenue of gas sold as input in later period. At the first period, each company uses the revenue of gas sold in previous period as one of the inputs. The chosen input and output normalized data for two six-month period that are used in the application are displayed in Table 1 and 2 as follows:

Table 1
The normalized data used in period 1

Companies	Budget	Number of staff	Rev. of gas sold in prev. period.	Amount of piping	Number of new cust.	Amount of branch-line	Rev. of gas sold in current period
# 1	0.9625	0.8665	0.9992	1	0.3352	0.4594	0.9398
# 2	0.9265	1	0.9969	0.569	0.1373	0.2048	1
# 3	1	0.9863	1	0.357	0.2617	0.5631	0.9907
# 4	0.6009	0.4059	0.8902	0.5915	0.8509	0.5466	0.8996
# 5	0.6617	0.7322	0.6873	0.937	0.682	0.8381	0.5277
# 6	0.5464	0.6271	0.4119	0.2558	0.1846	0.4144	0.4064
# 7	0.7287	0.628	0.5972	0.5177	0.7247	0.7187	0.7782
# 8	0.4038	0.1389	0.1789	0.487	0.6319	0.5499	0.9415
# 9	0.6186	0.4516	0.3959	0.3662	0.6799	0.5793	0.6134
# 10	0.7309	0.4598	0.3239	0.8213	1	1	0.7324
# 11	0.8250	0.7135	0.9957	0.1235	0.2304	0.223	0.5191

Table 2
The normalized data used in period 2

Companies	Budget	Number of staff	Rev. of gas sold in prev. period	Amount of piping	Number of new cust.	Amount of branch-line	Rev. of gas sold in current period
# 1	0.8973	0.9698	0.9398	1	0.3077	0.474	0.1878
# 2	0.3884	0.9943	1	0.5325	0.4978	0.3953	0.8419
# 3	0.7864	1	0.9907	0.2555	0.2935	0.354	1
# 4	0.6879	0.7926	0.8996	0.9130	1	0.9919	0.3372
# 5	1	0.7082	0.5277	0.9385	0.8206	0.5763	0.5516
# 6	0.9662	0.6008	0.4064	0.2656	0.3473	0.2137	0.3555
# 7	0.8261	0.6131	0.7782	0.5658	0.5917	0.5922	0.1811
# 8	0.9169	0.9416	0.9415	0.4614	0.4863	0.4912	0.9952
# 9	0.6223	0.4477	0.6134	0.3408	0.6628	0.3208	0.5262
# 10	0.8813	0.7639	0.7324	0.8819	0.979	1	0.4786
# 11	0.8876	0.9870	0.5191	0.7945	0.6105	0.5994	0.7394

By using sign of δ_t^* we measure type of return to scale in two periods and results are listed in Table 3.

Table 3
Return to scale in two period

Companies	δ_t^* in the first period		δ_t^* in the second period	
# 1	10.3107	(DRS)	16.9656	(DRS)
# 2	45.2381	(DRS)	0	(CRS)
# 3	53.8616	(DRS)	0	(CRS)
# 4	17.1814	(DRS)	0	(CRS)
# 5	0	(CRS)	0	(CRS)
# 6	0	(CRS)	0	(CRS)
# 7	5.4829	(DRS)	0	(CRS)
# 8	1.7030	(DRS)	5.4869	(DRS)
# 9	-6.8900	(IRS)	0	(CRS)
# 10	0	(CRS)	0	(CRS)
# 11	1.2730	(DRS)	0	(CRS)

6. Conclusions

In this study we use quasi-fixed inputs in dynamic data envelopment analysis. The quasi-fixed inputs are considered as outputs at the current period, while being treated as inputs at the next period. We definded dynamic Additive model then type of return to scale at the tth period identify by examinig sign of intercept of a supporting hyperplane. As a future extension of this research we can ranking DMUs in dynamic DEA in the each period.

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Received: September 20, 2007