Applied Mathematical Sciences, Vol. 2, 2008, no. 30, 1487 - 1491

Computing of Signs of Eigenvalues for Diagonally Dominant Matrix for Inertia Problem

F. Naeimi Dafchahi

Department of Mathematics, Faculty of Sciences
University of Guilan, P.O. Box 41335-1914, Rasht, Iran
f_naeimidafchahi@yahoo.com

Abstract

In this paper we present two theorems for computing of inertia of Diagonally Dominant Matrix and show a good relation between the inertia theorem and the signs of diagonal entries of the diagonally dominant matrix.

Mathematics Subject Classification: 15A18

Keywords: Eigenvalue, Inertia, Diagonally Dominant Matrix

1 Introduction

we know a system

$$\dot{x}(t) = Ax(t) \tag{1}$$

is a asymptotically stable, i.e., $x(t) \to 0 \quad as \quad t \to \infty$ if and only if all eigenvalues of A have negative real parts see [1].The stability of the system (1) is one of the most important problems in sciences and engineering. Lyapunov in 1892 in his PhD thesis described the matrix equation $XA + A^*X = -M$ and also discussed the stability of system (1)

2 Preliminary Notes

Definition 2.1 *Inertia of a matrix [2]*

The inertia of a $n \times n$ complex matrix A is shown by In (A) and is defined to be an integer triple $In(A)$ =$(\pi(A), \upsilon(A), \delta(A))$ where $\pi(A)$ is the number of eigenvalues of A with positive real parts.$\upsilon(A)$ and $\delta(A)$ with negative and zero real parts respectively.

3 Main Results

These are the main results of the paper.

Theorem 3.1 *The diagonal entries of a diagonally dominant matrix are not zero [2].*

Theorem 3.2 *The diagonally dominant matrix is invertible [2].*

Theorem 3.3 *(Ostrowski and Schneider-The main inertia theorem) [1, 3, and 4]:*

i)A necessary and sufficient condition for existing a symmetric matrix X where $XA + A^tX$ be symmetric and positive definite is that $\delta(A) = 0$.

ii)In(A)=In(X).

Theorem 3.4 *i)If $A = [a_{ij}]$ is a real row and column diagonally dominant matrix then there exist a symmetric matrix X such that $In(A) = In(X)$.*

ii)$In(A) =$(The number of positive diagonal entries; the number of negative diagonal entries; 0)

Proof. By theorem 3.2, we have $\delta(A) = 0$ and from theorem 3.3 there is a symmetric matrix X such that $XA + A^tX$ is symmetric and positive definite, and $In(A) = In(X)$.

Now let $X = diag(sgn(a_{ii}))$, $sgn(a_{ii}) = \begin{pmatrix} 1 & a_{ii} > 0 \\ -1 & a_{ii} < 0 \end{pmatrix}, i = 1...n$

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

Then we have

$$XA = \begin{pmatrix} |a_{11}| & (sgn(a_{11}))a_{12} & . & . & (sgn(a_{11}))a_{1n} \\ (sgn(a_{22}))a_{21} & |a_{22}| & . & . & (sgn(a_{22}))a_{2n} \\ . & (sgn(a_{33}))a_{32} & . & . & (sgn(a_{33}))a_{3n} \\ . & . & . & . & . \\ (sgn(a_{nn}))a_{n1} & . & . & (sgn(a_{nn}))a_{n,n-1} & |a_{nn}| \end{pmatrix}$$

$$A^tX = \begin{pmatrix} |a_{11}| & (sgn(a_{22}))a_{21} & . & . & (sgn(a_{nn}))a_{n1} \\ (sgn(a_{11}))a_{12} & |a_{22}| & . & . & (sgn(a_{nn}))a_{n2} \\ . & (sgn(a_{22}))a_{23} & . & . & (sgn(a_{nn}))a_{n3} \\ . & . & . & . & . \\ (sgn(a_{11}))a_{1n} & . & . & (sgn(a_{n-1,n-1}))a_{n-1,n} & |a_{nn}| \end{pmatrix}$$

If $XA + A^tX = [c_{ij}]\,, i = 1...n, j = 1..n$
Then:

$$c_{ij} = \begin{pmatrix} 2\,|a_{ii}| & i = j \\ (sgn(a_{ii}))a_{ij} + (sgn(a_{jj}))a_{ji} & i \neq j \end{pmatrix}$$

With respect to the above matrix and A is a row and column diagonally dominant we have

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |\, c_{ij} \,| = \sum_{\substack{j=1 \\ j \neq i}}^{n} |(sgn(a_{ii}))a_{ij} + (sgn(a_{jj}))a_{ji}| \leq$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}| < 2\,|a_{ii}| = |c_{ii}|\,, i = 1..n$$

Therefore$XA+A^tX$ is a diagonally dominant matrix with positive diagonal entries, so it is positive definite matrix. Then $In(A) = In(X)$ and since X is a diagonal matrix so;

$In(A)$ =(The number of positive diagonal entries; the number of negative diagonal entries; 0)

Theorem 3.5 (3, 4) *i)A necessary and sufficient condition for a hermitian matrix X to exist such that $XA + A^*X$ is hermitian and positive definite is that$\delta(A) = 0$.*

i)In(A)=In(X).

Theorem 3.6 *i)If $A = [a_{ij}]$ is a complex row-column diagonally dominant and$a_{ii} \in R$, then there exist a hermitian matrix X such that$In(A) = In(X)$.*

ii)$In(A)$ =(The number of positive diagonal entries; the number of negative diagonal entries; 0)

Proof. Since A is a diagonally dominant matrix so according to theorem 3.2 we have $\delta(A) = 0$ and by theorem 3.5 there exist a hermitian matrix Xsuch that $XA + A^*X$ is hermitian positive definite and $In(A) = In(X)$.

Now if we set $X = diag(sgn(a_{ii})), i = 1...n$ then by a similar proof as in 3.4 we conclude that X is a desired matrix,$In(A) = In(X)$.

Since X is diagonal so:

$In(A)$ =(The number of positive diagonal entries; the number of negative diagonal entries; 0)

3.7 Example

Consider the Sturm-Liouville problem as:

$$\begin{array}{l} \frac{d^2y}{dx^2} + [q(x) + \eta r(x)]y = 0 \\ a < x < b \\ y(a) = 0 \\ y(b) = 0 \end{array}$$

The object in this problem is findingη.
During solving this problem we obtain the matrix

$$A = \begin{pmatrix} a_{11} & -a_{12} & 0 & . & . & 0 \\ -a_{21} & a_{22} & -a_{23} & 0 & . & 0 \\ 0 & -a_{32} & a_{33} & . & . & 0 \\ . & 0 & . & . & . & . \\ . & . & . & . & . & -a_{n-1,n} \\ 0 & 0 & 0 & . & -a_{n,n-1} & a_{nn} \end{pmatrix}$$

$$a_{kk} = \frac{2 - h^2 q_k}{h^2 r_k}, a_{ij} = a_{ji} = \frac{1}{(r_i r_j)^{1/2}} \; |j - i| = 1$$

Where $\begin{array}{l} x_0 = a, x_k = x_0 + hk, \\ r(x_k) = r_k > 0 \\ q(x_k) = q_k \\ h = (b - a)/n + 1 \end{array}$

A is a diagonally dominant matrix and by choosing h as $|h| > (q_k/2)^{1/2}$ we have $a_{kk} < 0, k = 1...n$ then $In(A) = (0, n, 0)$.

3.8 Example

Consider Laplace equation as

$$\begin{array}{l} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad in \qquad R = \{(x, y) \,|\, 0 \le x \le \alpha, 0 \le y \le \beta\} \\ u(x, y) = g(x, y) \quad on \qquad S \end{array}$$

If we approximate this equation by finite difference method with $\Delta x = \frac{\alpha}{M_x} = \Delta y = \frac{\beta}{M_y}$ where $M_x = 4, M_y = 3$ then we obtain:

$$A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$

This matrix is also diagonally dominant with all positive diagonal entries, so by theorem 3.4 the signs of real parts of all eigenvalues of A are positive.

Conclusions. As the results show, without computing the eigenvalues of a diagonally dominant matrix, the location and the signs of the real parts of eigenvalues can be easily determined, just by observing the main diagonal entries; therefore we are able to discuss about the stability of the system.

References

[1] B.N.Datta, Stability and Inertia,Linear Algebra and its Appl **302-303** (1999), 563-600.

[2] B.N.Datta, Numerical Linear Algebra and Applications,Pacific Grove: California(1999).

[3] A.Ostrowski,H.Schneider, Some theorems on the inertia of general matrices ,J Math Anal Appl**4** (1962), 72-84.

[4] B.E.Cain, Inertia theory, Linear Algebra and its Appll**30** (1890), 211-240.

Received: February 20, 2008