Applied Mathematical Sciences, Vol. 2, 2008, no. 31, 1529 - 1539

On the Dynamic Behavior of a Delayed IS-LM Business Cycle Model

Abdelilah Kaddar and Hamad Talibi Alaoui [1]

Université Chouaib Doukkali, Faculté des Sciences
Département de Mathématiques et Informatique
B.P:20, El Jadida, Morocco

Abstract

In this paper, we formulate a delayed IS-LM model of business cycle. This model is represented by the Gabisch model [2] in considering the Kalecki assumption on time lag investment [5], i.e. there is a time shift after which capital equipment is available for production. A similar idea has been proposed by J. Cai [1], but the main difference with our model is the inclusion of the time delay into capital stock in capital accumulation equation. The dynamics are studied in terms of local stability and of the description of local Hopf bifurcation, that is proven to exist as the delay (taken as a parameter of bifurcation) cross some critical value. Additionally we conclude with an application.

Mathematics Subject Classification: 37G15, 91B62

Keywords: IS-LM business cycle, delayed differential equations, Hopf bifurcation, periodic solutions

1 Introduction

Kalecki (in 1935, [5]) was probably the first economist to introduce time delay in business cycle model, which is the result of time interval required between investment decision and installation of investment capital.
Besides the influence of Kyenes (in 1936, [10]) and Kalecki (in 1937, [6]), Kaldor (in 1940, [4]) proposed his first nonlinear business cycle model by an ordinary differential equations as follows

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{dK}{dt} = I(Y(t), K(t)), \end{cases}$$

[1] talibi_1@hotmail.fr

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, $I(Y,K)$ is the investment function and $S(Y,K)$ is the saving function.

In (1977, [12]) Torre revised and updated this model by replacing the capital stock $K(t)$ with the interest rate $R(t)$ to formulate the following standard IS-LM business cycle model

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t),R(t)) - S(Y(t),R(t))], \\ \frac{dR}{dt} = \beta[L(Y(t),R(t)) - \widetilde{M}], \end{cases}$$

where $\widetilde{M}$ is the constant money supply, β is the adjustment coefficient in money market and L is the demand for money.

In (1989, [2]), Gabisch and Lorenz considered an augmented IS-LM business cycle model as follows

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t),K(t),R(t)) - S(Y(t),R(t))], \\ \frac{dK}{dt} = I(Y(t),K(t),R(t)) - \delta K(t), \\ \frac{dR}{dt} = \beta[L(Y(t),R(t)) - \widetilde{M}], \end{cases}$$

where δ is the depreciation rate of capital stock.

Based on the Kalecki's idea of time delay (see [5, 8] for more information), Cai (in 2005, [1]) presented the following delayed IS-LM model:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t),K(t),R(t)) - S(Y(t),R(t))], \\ \frac{dK}{dt} = I(Y(t-\tau),K(t),R(t)) - \delta K(t), \\ \frac{dR}{dt} = \beta[L(Y(t),R(t)) - \widetilde{M}], \end{cases} \tag{1}$$

with τ is the time delay needed for new capital to be installed, and he investigated the local stability and the local Hopf bifurcation for (1) in the linear case.

In this paper, we think that it's more interesting to introduce the delay τ into gross product, capital stock and interest rate, because the change in the capital stock is due to the past investment decisions (see [9], p103). Thus in the following analysis we will consider the following delayed IS-LM business cycle model:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t),K(t),R(t)) - S(Y(t),R(t))], \\ \frac{dK}{dt} = I(Y(t-\tau),K(t-\tau),R(t-\tau)) - \delta K(t), \\ \frac{dR}{dt} = \beta[L(Y(t),R(t)) - \widetilde{M}], \end{cases} \tag{2}$$

The dynamics are studied in terms of local stability and of the description of the Hopf bifurcation, that is proven to exist as the delay (taken as a parameter of bifurcation) cross some critical value. In the end, we give some numerical simulations which show the existence and the nature of the periodic solutions.

2 Steady state and local stability analysis

As in Cai (2005, [1]), we assume that the investment function I, the saving function S, and the demand for money L are given by

$$I(Y,K,R) = \eta Y - \delta_1 K - \beta_1 R,$$

$$S(Y,R) = l_1 Y + \beta_2 R,$$

and

$$L(Y,R) = l_2 Y - \beta_3 R,$$

with $\delta_1, l_1, l_2, \beta_1, \beta_2, \beta_3$ are positive constants. Then system (2) becomes:

$$\begin{cases} \frac{dY}{dt} = \alpha[(\eta - l_1)Y(t) - \delta_1 K - (\beta_1 + \beta_2)R(t))], \\ \frac{dK}{dt} = \eta Y(t-\tau) - \delta_1 K(t-\tau) - \delta K(t) - \beta_1 R(t-\tau), \\ \frac{dR}{dt} = \beta[l_2 Y(t) - \beta_3 R(t) - \widetilde{M}]. \end{cases} \tag{3}$$

In the following proposition, we give a sufficient conditions for the existence and uniqueness of positive equilibrium E^* of the system (3).

Proposition 2.1 *Define*

$$\Theta = \delta(\beta_3\eta - \beta_1 l_2) - (\delta + \delta_1)(\beta_2 l_2 + \beta_3 l_1),$$

and suppose that
(H1): $\Theta < 0$;
(H2): $(\delta + \delta_1)l_1 - \delta\eta \leq 0$.
Then there exists a unique positive equilibrium $E^* = (Y^*, K^*, R^*)$ *of system (3), where* Y^*, K^*, R^* *are given by*

$$Y^* = \frac{-((\beta_1 + \beta_2)\delta + \beta_2\delta_1)\widetilde{M}}{\Theta}, \tag{4}$$

$$K^* = \frac{-(\beta_1 l_1 + \beta_2\eta)\widetilde{M}}{\Theta}, \tag{5}$$

and

$$R^* = \frac{((\delta + \delta_1)l_1 - \delta\eta)\widetilde{M}}{\Theta}. \tag{6}$$

Proof.
(Y, K, R) is a steady-state of (3) if

$$\frac{dY}{dt} = \frac{dK}{dt} = \frac{dR}{dt} = 0,$$

that is

$$\left\{\begin{array}{c}(\eta - l_1)Y - \delta_1 K - (\beta_1+\beta_2)R = 0,\\ \eta Y - (\delta+\delta_1)K - \beta_1 R = 0,\\ l_2 Y - \beta_3 R - \widetilde{M} = 0.\end{array}\right. \tag{7}$$

We have

$$\begin{aligned}\det\begin{pmatrix}\eta - l_1 & -\delta_1 & -(\beta_1+\beta_2)\\ \eta & -(\delta+\delta_1) & -\beta_1\\ l_2 & 0 & -\beta_3\end{pmatrix} &= \delta(\beta_3\eta - \beta_1 l_2) - (\delta+\delta_1)(\beta_2 l_2 + \beta_3 l_1)\\ &= \Theta.\end{aligned} \tag{8}$$

In view of hypotheses (H1) and (H2) of proposition 2.1 it's clear that system (7) has a unique positive solution given by (4), (5) and (6).

In the next, we will study the stability of the positive equilibrium E^* with respect to the time delay.
The characteristic equation associated to system (9) takes the general form

$$P(\lambda) + Q(\lambda)exp(-\lambda\tau) = 0 \tag{9}$$

with

$$P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$$

and

$$Q(\lambda) = D\lambda^2 + E\lambda + F,$$

where

$$A = \delta + \beta\beta_3 - \alpha(\eta - l_1),$$
$$B = \alpha\beta l_2(\beta_1+\beta_2) + \beta\beta_3\delta - \alpha(\delta+\beta\beta_3)(\eta - l_1),$$
$$C = \alpha\beta\delta[(\beta_1+\beta_2)l_2 - \beta_3(\eta - l_1)],$$
$$D = \delta_1,$$
$$E = \delta_1(\beta\beta_3 + \alpha l_1),$$

and

$$F = \alpha\beta\delta_1(\beta_2 l_2 + \beta_3 l_1).$$

Recall that the equilibrium of (3) is asymptotically stable if all roots of (9) have negative real parts, and the stability is lost only if characteristic roots cross the imaginary axis, that is if pure imaginary roots appear. In order to investigate the local stability of the steady state, we begin by considering the case without delay $\tau = 0$. This case is of importance, because it can be necessary that the nontrivial positive equilibrium of (3) is stable when $\tau = 0$

to be able to obtain the local stability for all nonnegative values of the delay, or to find a critical values which could destabilize the equilibrium.
When $\tau = 0$ the characteristic equation (9) reads as

$$\lambda^3 + (A + D)\lambda^2 + (B + E)\lambda + (C + F) = 0. \tag{10}$$

From (H1) we have $C + F > 0$. Hence, according to the Routh-Hurwitz criterion, we have the following,

Proposition 2.2 *For $\tau = 0$, the equilibrium E^* is locally asymptotically stable if and only if*
(H3): $A + D > 0$;
(H4): $(A + D)(B + E) - (C + F) > 0$;
where A, B, C, D, E, are defined in (9).

We assume in the sequel, that hypotheses (H1), (H2), (H3) and (H4) are true, and we return to the study of equation (9) with $\tau > 0$. Clearly, $\lambda(\tau) = u(\tau) + iv(\tau)$ is a root of equation (9) if and only if

$$u^3 - 3uv^2 + Au^2 - Av^2 + Bu + C = -\exp(-u\tau)\{Du^2\cos(v\tau)$$

$$-Dv^2\cos(v\tau) + Eu\cos(v\tau) + F\cos(v\tau) + 2Duv\sin(v\tau) + Ev\sin(v\tau)\}, \tag{11}$$

and

$$3u^2v - v^3 + 2Auv + Bv = -\exp(-u\tau)\{2Duv\cos(v\tau)$$

$$+Ev\cos(v\tau) - Du^2\sin(v\tau) + Dv^2\sin(v\tau) - Eu\sin(v\tau) - F\sin(v\tau)\}, \tag{12}$$

We set $u = 0$ into the two equation (11) and (12) to get

$$-Av^2 + C = (Dv^2 - F)\cos(v\tau) - Ev\sin(v\tau), \tag{13}$$

and

$$v^3 - Bv = Ev\cos(v\tau) + (Dv^2 - F)\sin(v\tau). \tag{14}$$

Squaring and adding the squares together, we obtain

$$v^6 + av^4 + bv^2 + c = 0, \tag{15}$$

with $a = A^2 - D^2 - 2B$, $b = B^2 - 2AC - E^2 + 2DF$, $c = C^2 - F^2$, where $A; B; C; D; E$ are given by (9).
Letting $z = v^2$, equation (15) becomes the following cubic equation

$$h(z) := z^3 + az^2 + bz + c = 0, \tag{16}$$

Lemma 2.3 *[11] Define*

$$\Delta = a^2 - 3b, \tag{17}$$

(i) If $c < 0$, then equation (16) has at least one positive root.
(ii) If $c \geq 0$ and $\Delta \leq 0$, then equation (16) has no positive roots.
(iii) If $c \geq 0$ and $\Delta > 0$, then equation (16) has positive roots if and only if $\overline{z} := \frac{1}{3}(-a + \sqrt{\Delta}) > 0$ and $h(\overline{z}) \leq 0$.

Suppose that equation (16) has positive roots. Without loss of generality, we assume that it has three positive roots, denoted by z_1, z_2 and z_3, respectively. Then equation (15) has three positive roots, say

$$v_1 = \sqrt{z_1}; v_2 = \sqrt{z_2}; v_3 = \sqrt{z_3}$$

Let

$$\tau_l^j = \frac{1}{v_l}[\arccos(\frac{(Av_l^2 - C)(F - Dv_l^2) + (v_l^3 - Bv_l)Ev_l}{(Dv_l - F)^2 + E^2 v_l^2}) + 2j\pi], l = 1, 2, 3; j = 0, 1....$$

Then $\pm iv_l$ is a pair of purely imaginary roots of equation (9) with $\tau = \tau_l^j$, l=1,2,3; j=0,1.... Clearly,

$$\lim_{j\to\infty} \tau_l^j = \infty, l = 1, 2, 3.$$

Thus, we can define

$$\tau_0 = \tau_{l_0}^{j_0} = \min_{j=0,1...,l=1,2,3}(\tau_l^j), v_0 = v_{l_0}. \tag{18}$$

From proposition 2.2 and lemma 2.3, we have the following lemma.

Lemma 2.4 *Suppose that (H1)-(H4) hold.*
(i) If one of the following:
(N1) $c \geq 0$ and $\Delta \leq 0$;
(N2) $c \geq 0$ $\Delta > 0$, and $\overline{z} \leq 0$;
(N3) $c \geq 0$ $\Delta > 0$, $\overline{z} > 0$, and $h(\overline{z}) \leq 0$;
is true, then all roots of equation (9) have negative real parts for all $\tau \geq 0$.
(ii) If $c < 0$, or $c \geq 0$, $\Delta > 0$, $\overline{z} > 0$ and $h(\overline{z}) \leq 0$, then all roots of equation (9) have negative real parts when $\tau \in [0, \tau_0)$,
where Δ and $\overline{z} > 0$ are defined in lemma 2.1.

Next we need to guarantee the transversality condition of the Hopf bifurcation theorem (see [3]). Let $\lambda(\tau) = u(\tau) + iv(\tau)$ be the root of equation (9) satisfying $u(\tau_0) = 0$, and $v(\tau_0) = v_0$.

Lemma 2.5 *Suppose that (H1)-(H4) hold.*
If one of the following:
(S1) $c < 0$, *and* $h'(v_0^2) \neq 0$;
(S2) $c \geq 0$, $\Delta > 0$, $\overline{z} > 0$ *and* $h(\overline{z}) < 0$;
is true, then

$$\frac{dRe\lambda(\tau_0)}{d\tau} > 0,$$

where τ_0, *and* v_0 *are defined in (18).*

Proof

By differentiating equations (11) and (12) with respect to τ and then set $\tau = \tau_0$. Doing this, we get

$$G_1 \frac{du(\tau_0)}{d\tau} + G_2 \frac{dv(\tau_0)}{d\tau} = H_1, \tag{19}$$

$$-G_2 \frac{du(\tau_0)}{d\tau} + G_1 \frac{dv(\tau_0)}{d\tau} = H_2, \tag{20}$$

where

$$G_1 = -3v_0^2 + B + (E + Dv_0^2\tau_0 - F\tau_0)\cos(v_0\tau_0) + (2Dv_0 - Ev_0\tau_0)\sin(v_0\tau_0),$$

$$G_2 = -2Av_0 + (-2Dv_0 + Ev_0\tau_0)\cos(v_0\tau_0) + (E + Dv_0^2\tau_0 - F\tau_0)\sin(v_0\tau_0),$$

$$H_1 = (-Dv_0^3 + Fv_0)\sin(v_0\tau_0) - Ev_0^2\cos(v_0\tau_0),$$

and

$$H_2 = (-Dv_0^3 + Fv_0)\cos(v_0\tau_0) + Ev_0^2\sin(v_0\tau_0).$$

Solving for $\frac{du(\tau_0)}{d\tau}$ we get

$$\frac{du(\tau_0)}{d\tau} = \frac{G_1H_1 - G_2H_2}{G_1^2 + G_2^2}, \tag{21}$$

Therefore, we have

$$\frac{du(\tau_0)}{d\tau} = \frac{v_0^2 h'(v_0^2)}{G_1^2 + G_2^2}, \tag{22}$$

Note that if $h(\overline{z}) < 0$, then $h'(v_0^2) \neq 0$, because $h(\pm\infty) = \pm\infty$ and $h(0) = c \geq 0$.
Thus, if $h'(v_0^2) \neq 0$ we have the transversality condition:

$$\frac{du(\tau_0)}{d\tau} \neq 0.$$

If $\frac{du(\tau_0)}{d\tau} < 0$ for $\tau < \tau_0$ and close to τ_0, then equation (9) has a root $\lambda(\tau) = u(\tau) + iv(\tau)$ satisfying $u(\tau) > 0$, which contradicts (ii) of lemma 2.4. This completes the proof.

By lemmas 2.4 and 2.5, we obtain the following theorem.

Theorem 2.6 *Assume that (H1)-(H4) hold,*
(a) If (i) of lemma 2.4 holds, then, the equilibrium E^* *of system (3) is locally asymptotically stable for all* $\tau \geq 0$.
(b) If (S1) or (S2) in lemma 2.5 holds. then there exists a positive τ_0 *such that, when* $\tau \in [0, \tau_0)$ *the steady state* E^* *is locally asymptotically stable, and a Hopf bifurcation occurs as* τ *passes through* τ_0*, where* τ_0 *is given by*

$$\tau_0 = \frac{1}{v_0} \arccos \frac{(Av_0^2 - C)(F - Dv_0^2) + (v_0^3 - Bv_0)Ev_0}{(Dv_0 - F)^2 + E^2 v_0^2}, \qquad (23)$$

and v_0 *is the least simple positive root of equation (12), with* A, B, C, D, E, *are defined in (9).*

3 Application

Proposition 3.1 *If*

$$\alpha = 0.96; \beta = 2; \delta = 0.2; \delta_1 = 0.5; \beta_1 = \beta_2 = \beta_3 = 0.2; l_1 = l_2 = 0.1; \eta = 0.4; \widetilde{M} = 0.05.$$

Then systems (4) have the following positive equilibrium

$$E^* = (0.5624, 0.3125, 0.03125).$$

Furthermore, the critical delay corresponding to (3) is $\tau_0 = 2.437523028$.

By theorem 2.1 and proposition 3.1, we have if $\tau < 2.437523028$, then E^* is locally asymptotically stable (see Fig.1). If we increase the value of τ, then a periodic solution occurs at $\tau_0 = 2.437523028$ (see Fig.2) and E^* becomes unstable for $\tau > 2.437523028$ (see Fig.3).

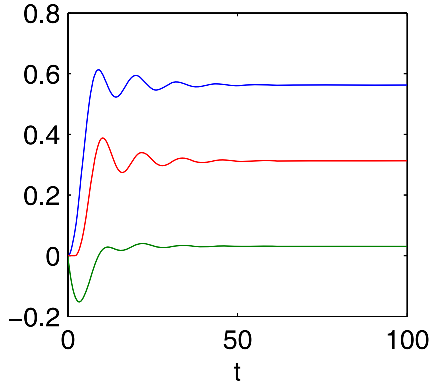

Figure 1: For $\tau = 2$ solutions $Y(t)$ (blue line), $R(t)$ (green line), $K(t)$ (red line) of (3) are asymptotically stable and converge to the equilibrium E^*.

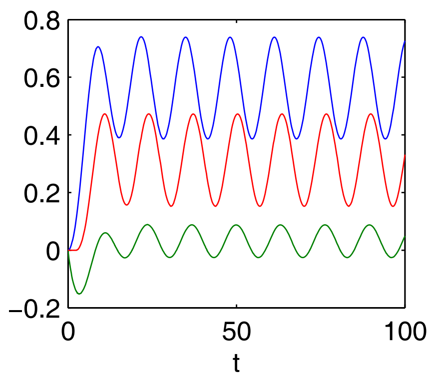

Figure 2: When $\tau = 2.4375$, a Hopf bifurcation occurs and periodic solutions appear, with same period for the three solutions $Y(t)$ (blue line), $R(t)$ (green line), $K(t)$ (red line) of (3).

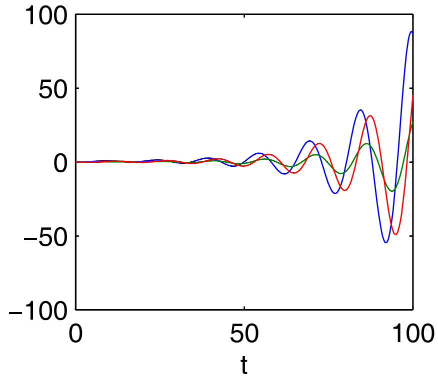

Figure 3: The steady state E^* of (3) is unstable when $\tau = 3$.

References

[1] J.P. Cai, Hopf bifurcation in the IS-LM business cycle model with time delay, Electronic Journal of Differential Equations, 2005(15):1-6.

[2] G. Gabisch and H.W. Lorenz, (1987) Business Cycle Theory: A survey of methods and concepts. 1989 edition Berlin:Springer-Verlag.

[3] J. K. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer- Verlag, New York, 1993.

[4] N. Kaldor, A Model of the Trade Cycle, Economic Journal, 1940, V.50, 78-92.

[5] M. Kalecki, A Macrodynamic Theory of Business Cycles, Econometrica, 1935, V.3, 327-344.

[6] M. Kalecki, A Theory of the Busines cycle, Rev. Studies 4 (1937), 77-97.

[7] Q. J. A. Khan, Hopf bifurcation in multiparty political systems with time delay in switching, Applied Mathematics Letters, 13(2000), 43-52.

[8] A. Krawiec and M. Szydlowski, The Kaldor-Kalecki Business Cycle Model, Ann. of Operat. Research, 1999, V.89, 89-100.

[9] A. Krawiec and M. Szydlowski, On nonlinear mechanics of business cycle model, Regular and Chaotic Dynamics, V.6, N.1,2001.

[10] J.M. Kynes, The General Theory of Employment, Interest Money, Macmillan Combridge University Press, 1936.

[11] S. Ruan, J. Wei, On the zeros of a third degree exponential polynomial with applications to a delayed model for the control of testosterone secretion. IMA J. Math. Appl. Med. Biol. 18, 41?52,2001.

[12] V. Torre, Existence of limit cycles and control in complete Kynesian systems by theory of bifurcations, Econometrica, 45(1977), 1457-1466.

Received: December 7, 2007