Applied Mathematical Sciences, Vol. 2, 2008, no. 31, 1541 - 1548

A Note on One-Dimensional Optimal System of Generalized Boussinesq Equation

Figen Açil Kiraz

Department of Mathematics
Univrsity of Balikesir
Faculty of Art and Sciences
101045 Balikesir, Turkey
figen.acil.kiraz@hotmail.com

Abstract

In this paper Lie group classification of a Generalized Bonussinesq Equation (p, q, r are constant) is obtained by the one-parameter optimal system of one-dimensional subalgebras of the Lie Algebra which is one-to-one correspondence with the Lie group and their associated reduced equations.

Mathematics Subject Classification: 22E70, 35A30, 58J70

Keywords: Lie group, symmetry, generalized boussinesq equation, optimal system, invariant, adjoint representation.

1 Introduction

In applied group analysis, Lie theory of symmetry group for differential equations, constituted by Sophus Lie , is the most important solution method for the nonlinear problems in the field of applied maths. The fundamentals of Lie's theory are based on the invariance of the equation under transformation groups of independent and dependent variables, so called Lie groups. In the last century, the application of the Lie group method has been developed by a number of mathematicians. Ovsiannikov [6], Olver [14], Ibragimov [8], Baumann [2] and Bluman and Anco [5] are some of the mathematicians who have enormous amount of studies in this field.

The existence of symmetries of diferential equations under Lie group of transformations often allows those equations to be reduced to simpler equations. One of the major accomlishment of Lie was to identify that the properties of global transformations of the group are completely and uniquely determined by the infinitesimal transformations around the identity transformation.

This allows the nonlinear relations for the identification of invariance groups to be dealing with global transformation equations, we use differential operators, called the group generators, whose exponentiation generates the action of the group. The collection of these differentail operators forms the basis for the Lie algebra. There is a one-to-one correspondence between the Lie groups and the associated Lie algebras.

A basic problem concerning the group invariant solution is its classification. Since a Lie group (or Lie algebra) usually contain inifitely many subgroups (or subalgebras) of the same dimensional, a classification of them up to some equivalence relation is necessary. Ovsiannikov [6] given equivalent of two subalgebras of a given Lie algebra. Optimal system consists of representative elements of each equality class. Disussion on optimal systems can be found in [14], [6]. Some examples of optimal system can also be found in Ibragimov [9].

In this paper, we find one-dimensional optimal system for equation (1)(p,q,r are constant) and classify reductions obtained by using one-dimensional subalgebras.

2 Lie Point Symmeties

Consider, now, Generalised Boussinesq (GBQ) Equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 + u_{tt} = 0 \tag{1}$$

where p,q and r are constants such that $r \neq 0$ and subscripts denote partial derivatives.

Clasical symmetry reductions of some special cases of equation (1) have been discussed by Schwarz [1], Clarkson [12], Kawamoto [15], Lou [16], Paquin and Winternitz [4]. Clasical symmetrys of some different type of equation (1) have been investigated by Clarkson and Priestly [13], Gandarias and Bruzon [7]. Clarkson and Kruskal [10] developed a direct method (in the sequel referred as the Direct Method) for finding symmetry reductions which is used to obtain previously unknown reductions of the Boussinesq Equation and Clarkson and Ludlow [11] derived new nonclassical smmetry reductions of generalised Boussinesq equation by using Direct method and said that those derived by using the Lie group method with one illustration.

To apply the classical Lie group method to the GBQ equation (1), we perform symmetry analysis. Let us consider a one-parameter Lie group of infinitesimal transformation

$$x \to x + \varepsilon\xi_1(x,t,u) + O(\varepsilon^2)$$

$$t \to t + \varepsilon\xi_2(x,t,u) + O(\varepsilon^2) \tag{2}$$

$$u \to u + \varepsilon\eta(x,t,u) + O(\varepsilon^2)$$

where ε is group parameter in (x,t,u)-space. The vector field associated with the above group of transformations can be written as

$$U = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$$

This is symmetry generator and invariance of equation (1) under transformation (2). Solving the determining equations yields the following infinitesimals (throughout this paper we will use package MathLie to perform all calculation [2]):

p,q,r constant : $\xi_1 = k_4x + k_3$
$\xi_2 = 2k_4t + k_2$
$\eta = k_1$

p=0,q,r constant : $\xi_1 = k_5x + k_4$
$\xi_2 = 2k_5t + k_3$
$\eta = k_2t + k_1$

p≠0, r=$\frac{1}{4}q(p+q)$: $\xi_1 = k_5x + \frac{1}{2}k_3qt + k_4$,
$\xi_2 = 2k_5t + k_1$
$\eta = k_3x + k_2$

p=0, r=$\frac{1}{4}q^2$: $\xi_1 = k_6x + \frac{1}{2}k_4qt + k_5$
$\xi_2 = 2k_6t + k_2$
$\eta = k_4x + k_1t + k_3$

p=q, r=$\frac{1}{2}q^2$: $\xi_1 = k_6qtx + k_3x + \frac{1}{2}k_5qt + k_2$
$\xi_2 = k_6qt^2 + 2k_3t + k_1$
$\eta = k_6x^2 + k_5x + k_4$

where $k_1, k_2, k_3, k_4, k_5, k_6$ are arbitrary constants. The symmetry variables are then found by solving the characteristc equations

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\eta}$$

and then, substituting the resulting expression into (1), one obtains the reduced equation.

Clarkson and Ludlow obtained some similarity transformations of equation (1) with direct method and said that this similarity transformations can be obtained from Lie group method with one illustration in [11].

A basic problem concerning the group invariant solution is its classification. We cannot say anything concerning whether equation (1) is invariant under group transformation corresponding to similarity transformations which is obtained with the direct method. Furthermore, direct method cannot give the answer to the question about whether we can obtain other similarity transformations of this type, for we cannot specify a connection between similarity transformations obtained through the direct method.

In this paper it is showed that symmetry reductions obtained from the direct method of equation (1) (p,q,r constant) correspond symmetry reductions obtained from optimal system of one-dimensional subalgebras of Lie algebra which is of infinitesimal symmetries for this equation. Thus, we ensure that equation (1) is invariant under group transformation and all reductions which are performed with one-dimensional subalgebras of Lie algebra are find with optimal system.

3 A One-Parameter Optimal System

The construction of the one-paremeter optimal system of one-dimensional subalgebras can be made by using a global matrixof the adjoint transformations as suggested by Ovsiannikov [6]. In this paper we follow, instead, the method by Olver [14] which uses a slightly different technique. It consist in constructig a table, which is usually called the adjoint table, showing the separate adjoint actions of each element the lie algebra on all other elements.

In this section we will give an optimal system for equation (1) where p,q,r are constant. Let us take infinitesimals above

$$\begin{aligned}\xi_1 &= k_4x+k_3\\ \xi_2 &= 2k_4t+k_2\\ \eta &= k_1\end{aligned}$$

where k_1, k_2, k_3, k_4 are arbitrary constants. Hence, symmetry generator of equation (1) is

$$U=(k_4x+k_3)\frac{\partial}{\partial x}+(2k_4t+k_2)\frac{\partial}{\partial t}+k_1\frac{\partial}{\partial u}.$$

The presence of these arbitrary constants lead to a finite-dimensional Lie algebra of symmetries. A general element of this algebra is written as

$$X=a_1X_1+a_2X_2+a_3X_3+a_4X_4$$

where

$$X_1=\frac{\partial}{\partial u},\quad X_2=\frac{\partial}{\partial t},\quad X_3=\frac{\partial}{\partial x},\quad X_4=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial u}$$

construct a basis of vector space. The Lie algebra of infinitesimal symmetries for equation (1) is spanned by these base vectors. The associated Lie algebra among these vector fields becomes

[,]	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	$-2X_2$
X_3	0	0	0	$-X_3$
X_4	0	$2X_2$	X_3	0

where the entry in jth row and kth column represents the Lie product $[X_j,X_k]$ [14]. In order to find the optimal system of this equation, first the following adjoint table is constituted from the Lie product table and the definition of adjoint representation [14].

$Ad(\exp(\varepsilon^*)^*)$	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	$X_4+2\varepsilon_2X_2$
X_3	X_1	X_2	X_3	$X_4+\varepsilon_3X_3$
X_4	X_1	$e^{-2\varepsilon_4}X_2$	$e^{-\varepsilon_4}X_3$	X_4

Thus, the following matrixes are obtained by using adjoint table [3].

$$Ad(e^{\varepsilon_1 X_1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Ad(e^{\varepsilon_2 X_2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2\varepsilon_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Ad(e^{\varepsilon_1 X_1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Ad(e^{\varepsilon_1 X_1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\varepsilon_4} & 0 & 0 \\ 0 & 0 & e^{-\varepsilon_4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then adjoint representation of any element of the group is

$$Ad_g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\varepsilon_4} & 0 & 2\varepsilon_2 \\ 0 & 0 & e^{-\varepsilon_4} & \varepsilon_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is obtained by being multiplied with these matrixes and following equalities are found by using adjoint representation [3].

$$\frac{1}{a} Ad_g = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$\frac{1}{a}\begin{pmatrix} \alpha_1 \\ \alpha_2 e^{-2\varepsilon_4} + 2\alpha_4\varepsilon_2 \\ \alpha_3 e^{-\varepsilon_4} + \alpha_4\varepsilon_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

We are trying to simplify the right hand side of last equality by determining ε_i, where $i = 1, 2, 3, 4$. We have to distinguish several case referring to α_4.

1.$\alpha_4 \neq 0 : (\alpha_4 = a, \beta_4 = 1)$

We begin with the second and third component. $\beta_2 = \beta_3 = 0$ are obtained by choosing $\varepsilon_3 = -\frac{\alpha_3}{\alpha_4} e^{-\varepsilon_4}$ and $\varepsilon_2 = -\frac{\alpha_2}{2\alpha_4} e^{-2\varepsilon_4}$ and we find $\beta = (\lambda, 0, 0, 1)$with $\lambda = \frac{\alpha_1}{a} \in IR$. Thus, we obtain generator which is $\lambda X_1 + X_4$.

2.$\alpha_4 = 0$

Thus, we have

$$\frac{1}{a}\begin{pmatrix} \alpha_1 \\ \alpha_2 e^{-2\varepsilon_4} \\ \alpha_3 e^{-\varepsilon_4} \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

In this case ,for $\lambda = \frac{\alpha_1}{a} \in IR$, we have $\alpha_2 \neq 0$, $\alpha_3 \neq 0$ and $\alpha_2 \neq 0$, $\alpha_3 = 0$ and $\alpha_2 = 0$, $\alpha_3 \neq 0$ and $\alpha_2 = 0$, $\alpha_3 = 0$ and . if either of them different zero, than we find

$$\beta_2 = \begin{cases} 1 & \alpha_2 > 0 \\ -1 & \alpha_2 < 0 \end{cases},$$

by choosing $\varepsilon_4 = \frac{1}{2} \ln \frac{|\alpha_2|}{a}$and

$$\beta_3 = \begin{cases} 1 & \alpha_3 > 0 \\ -1 & \alpha_3 < 0 \end{cases}$$

by choosing $\varepsilon_4 = \ln \frac{|\alpha_2|}{a}$ from following equalitys $\beta_2 = -\frac{\alpha_2}{a} e^{-2\varepsilon_4}$, $\beta_3 = -\frac{\alpha_3}{a} e^{-2\varepsilon_4}$. Thus we find $\beta = (\lambda, \delta, \delta, 1)$with $\delta = \{-1, 0, 1\}$ and we obtain generator which is $\lambda X_1 + \delta X_2 + \delta X_3$.

Finally, the optimal system consist of

$$\lambda X_1 + X_4,\ \lambda X_1 + \delta X_2 + \delta X_3$$

where $\lambda \in IR, \delta \in \{-1, 0, 1\}$.

Now, let us do similarity reductions by using the optimal system.

1. Reduction by using algebra $L^{\lambda}_{1,1}$:

From equation $(\lambda X_1 + X_4\ )\ w(x, t, u) = 0$, similarity variables $z = \frac{x}{\sqrt{t}}, u = v(z) + \frac{\lambda}{2} \ln t$ is found by solving characteristic equation

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{\lambda}$$

If reduction of equation (1) is done by these similarity variables, then

$$W'' - \frac{1}{2}z(p+q)WW' + (\frac{\lambda}{2}p + \frac{1}{4}z^2)W' - \frac{q}{2}W^2 + rW^2W' + \frac{3}{4}zW - \frac{\lambda}{2} = 0.$$

where $v' = W$. if $q = 0$ and $r = -\frac{1}{2}p^2$ and we make the transformation

$$W(z) = \frac{1}{p}(-3^{\frac{3}{4}}y + z), \qquad x = -\frac{1}{2}3^{\frac{1}{4}}z$$

then $y(x)$ satisfies the fourth Painleve equation (PIV) [11].

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - A) + \frac{B}{y}$$

with $A = \frac{\lambda p}{6}$ and B a constant of integration.

2. Reduction by using algebra $L_{1,2}^{\lambda,\delta}$:

Similarity variables $z = x - t$, $u = v(z) + \frac{\lambda}{\delta}t = v(x-t) + \frac{\lambda}{\delta}t$ is obtained by solving equation $(\lambda X_1 + \delta X_2 + \delta X_3)\ w(x,t,u) = 0$.Thus, reduction of equation (1) done by these similarity variables is

$$W' + r\frac{W^3}{3} - (p+q)\frac{W^2}{2} + (\frac{\lambda}{\delta}p + 1)W = C$$

with C a constant of integration where $v' = W$. This equation is solved by using elliptic integral.

4 Concluding remarks

In this paper, we have determined an optimal system for Generalised Boussinesq (GBQ) equation (p,q,r are constant). Thus, one classification of the similarity solutions has been obtained. One reduction of equation (1) can be done by using two-dimensional subalgebras.

References

[1] F., Schwarz, *Automatically determining symmetries of partial differential equations* ,Computing, 34, 91-106 (1985).

[2] G., Baumann, Symmetry Analysis of Differential Equations with Mathematica. Telos, Springer Verlag, New York (2000).

[3] G., Baumann, J., Volkmann, *Connection between Painleve Analysis and Optimal Systems*, Dif. Eq.Cont.Pro. no.4 (2002).

[4] G., Paquin, P., Winternitz, *Group theoretical analysis of dispersive long wave equations in two space dimensions*. Phys. D 46 (1990), no. 1, 122-138.

[5] G.W., Bluman, S.C., Anco, Symmetry and Integtation Methods for Differential Equations, Springer-Verlag, New York (2002).

[6] L.V., Ovsyannikov, Group Analysis of Differential Equations, Academic Pres, New York, (1982).

[7] M.L., Gandaraias, M.S., Bruzon, *Classical and Nonclassical Symmetries of a Generalized Boyssinesq Equation*, J.Nonlinear Math. Phys.,V.5, N 1,8-12(1998).

[8] N.H., Ibragimov, N.H., Handbook of Lie Group Analysis of Differential Equations, Volume1,2,3 CRC Press, Boca Raton, Ann Arbor, London, Tokyo, (1994,1995,1996).

[9] N.H., Ibragimov, N.H., Transformation Groups Applied to Mathematical Physics. CRC Press, Boca Raton, F.L, (1985).

[10] P.A. Clarkson, M.D. Kruskal, *New similarity solutions of the Boussinesq equation*, J. Math. Phys., 30 (1989), 2201–2213.

[11] P.A., Clarkson, D.K.,Ludlow, *Symmetry Reductions, Exact Solutions and Painleve Analysis for a Gereralised Boussinesq Equation*, J. Math. Analysis and Appl.,186,132-155 (1994).

[12] P.A., Clarkson, *New Similarity Solutions for thr Modified Boussinesq Equation*, J.Phys. A.Math. Gen.,2355-2367(1998).

[13] P.A., Clarkson, T.J., Priestley, *Symmetry of a Generalized Boussinesq Equation*, Ins.Mat.and St. Un. Kent at Canterburg, CT2, 7NF, U.K. (1996).

[14] P. J.,Olver, Applications of Lie Groups to Differential Equations, GTM 107, Second edn., Springer-Verlag (1993).

[15] S., Kawamoto, *Linearization of the classical Boussinesq and related equations*. J.Phys. Soc. Japan, 53, N.9,2922-2929 (1984)

[16] S.Y., Lou, G.J., Ni, *Similarity analysis for dispersive wave equations in shallow water*, Comm. Theoret. Phys., 18, N.2, 165-170 (1992).

Received: February 27, 2008