Applied Mathematical Sciences, Vol. 2, 2008, no. 32, 1571 - 1580

A Simple Proof of Sylvester's (Determinants) Identity

Abdelmalek Salem

Department of Mathematics and Informatiques,
University Centre Cheikh Larbi Tebessi
of Tebessa 12002 Algeria
a.salem@gawab.com

Kouachi Said

University Centre of Khenchela 40100 Algeria
kouachi.said@caramail.com

Abstract

In this paper we give a simple proof of Sylvester Identity that is based on determinant properties and that is obtained inductively.

Mathematics Subject Classification: [2000]Primary 05A,15A

Keywords: Dodgson's Algorithm, Matrix, Determinant, Sylvester's Identity

1 Introduction

We can write the well-known algorithm of Dodgson, concerning a square matrix $A = (a_{i,j})_{1\leq i,j\leq n}$ as follows:

$$\det\left[(a_{i,j})_{1\leq i,j\leq n}\right]\det\left[(a_{i,j})_{\substack{i\neq k,l\\ j\neq k,l}}\right] = \det\begin{bmatrix} \det\left[(a_{i,j})_{\substack{i\neq l\\ j\neq l}}\right] & \det\left[(a_{i,j})_{\substack{i\neq l\\ j\neq k}}\right] \\ \det\left[(a_{i,j})_{\substack{i\neq k\\ j\neq l}}\right] & \det\left[(a_{i,j})_{\substack{i\neq k\\ j\neq k}}\right] \end{bmatrix}, \tag{1}$$

for all $k, l = 1, ..., n$ considering $k < l$. (see S.Kouachi, S.Abdelmalek and B.Rebai [4])

This formula enables us condense the determinant of n square matrix to the determinant of 2 square matrix. The elements of 2 square matrix are the determinants of $(n-1)$ square matrix.

In an other article (see S.Abdelmalek and S.Kouachi [1]), we condense the determinant of n square matrix to the determinant of $(n-1)$ square matrix. The elements of $(n-1)$ square matrix are the determinants of 2 square matrix. as it is shown in the following formula:

$$(a_{k,l})^{n-2} \det\left[(a_{i,j})_{1\leq i,j\leq n}\right] = \det_{1\leq i,j\leq n-1}\left[\det\left(A_{i,j}\right)\right], \qquad 1\leq k,\ l\leq n \tag{2}$$

when

$$A_{(i,j)} = \begin{cases} \begin{pmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{pmatrix} & \text{if } j<l, i<k \\ \begin{pmatrix} a_{i,l} & a_{i,j+1} \\ a_{k,l} & a_{k,j+1} \end{pmatrix} & \text{if } j\geq l, i<k \\ \begin{pmatrix} a_{k,j} & a_{k,l} \\ a_{i+1,j} & a_{i+1,l} \end{pmatrix} & \text{if } j<l, i\geq k \\ \begin{pmatrix} a_{k,l} & a_{k,j+1} \\ a_{i+1,l} & a_{i+1,j+1} \end{pmatrix} & \text{if } j\geq l, i\geq k \end{cases}.$$

We know that the generalization of formulas (1) and (2) lets us condense the determinant of any n square matrix to the determinant of m square matrix $(m<n)$. The elements of m square matrix are the determinants of $(n-(m-1))$ square matrix. And it is well-known by Sylvester's Identity in 1951 with no proof see [5] p-193 After him, others have given proofs see [2] we give an other clear proof.

We may clarify it by the following examples:

Example 1 *For example $n=7$ and $m=4$*

$$\left(\begin{vmatrix} 2 & 5 & 4 \\ 0 & 1 & 3 \\ 9 & 4 & 7 \end{vmatrix}\right)^{4-1} \begin{vmatrix} 2 & 5 & 4 & 7 & 6 & 1 & 2 \\ 0 & 1 & 3 & 8 & 8 & 1 & 5 \\ 9 & 4 & 7 & 8 & 9 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 2 & 0 & 8 \\ 11 & 2 & 5 & 4 & 5 & \frac{1}{2} & 5 \\ 5 & 7 & 8 & 6 & 1 & 0 & 5 \\ 9 & 2 & 3 & 5 & 8 & 5 & 3 \end{vmatrix} =$$

$$\begin{vmatrix}
\begin{vmatrix} 2 & 5 & 4 & 7 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 8 \\ 7 & 8 & 4 & \sqrt{3} \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 6 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 9 \\ 7 & 8 & 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & 1 & 3 & 1 \\ 9 & 4 & 7 & 8 \\ 7 & 8 & 4 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 2 \\ 0 & 1 & 3 & 5 \\ 9 & 4 & 7 & 6 \\ 7 & 8 & 4 & 8 \end{vmatrix} \\
\begin{vmatrix} 2 & 5 & 4 & 7 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 8 \\ 11 & 2 & 5 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 6 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 9 \\ 11 & 2 & 5 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & 1 & 3 & 1 \\ 9 & 4 & 7 & 8 \\ 11 & 2 & 5 & \frac{1}{2} \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 2 \\ 0 & 1 & 3 & 5 \\ 9 & 4 & 7 & 6 \\ 11 & 2 & 5 & 5 \end{vmatrix} \\
\begin{vmatrix} 2 & 5 & 4 & 7 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 8 \\ 5 & 7 & 8 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 6 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 9 \\ 5 & 7 & 8 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & 1 & 3 & 1 \\ 9 & 4 & 7 & 8 \\ 5 & 7 & 8 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 2 \\ 0 & 1 & 3 & 5 \\ 9 & 4 & 7 & 6 \\ 5 & 7 & 8 & 5 \end{vmatrix} \\
\begin{vmatrix} 2 & 5 & 4 & 7 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 8 \\ 9 & 2 & 3 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 6 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 9 \\ 9 & 2 & 3 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & 1 & 3 & 1 \\ 9 & 4 & 7 & 8 \\ 9 & 2 & 3 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 2 \\ 0 & 1 & 3 & 5 \\ 9 & 4 & 7 & 6 \\ 9 & 2 & 3 & 3 \end{vmatrix}
\end{vmatrix} \text{,}\textit{finally:}$$

$$(89)^3 \begin{vmatrix} 2 & 5 & 4 & 7 & 6 & 1 & 2 \\ 0 & 1 & 3 & 8 & 8 & 1 & 5 \\ 9 & 4 & 7 & 8 & 9 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 2 & 0 & 8 \\ 11 & 2 & 5 & 4 & 5 & \frac{1}{2} & 5 \\ 5 & 7 & 8 & 6 & 1 & 0 & 5 \\ 9 & 2 & 3 & 5 & 8 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 89\sqrt{3}-23 & 263 & -280 & 1005 \\ 270 & 194 & -\frac{1587}{2} & 163 \\ -718 & -1100 & -371 & -148 \\ 659 & 815 & -177 & 265 \end{vmatrix}.$$

Example 2 *For example $n = 7$ and $m = 3$*

$$\left(\begin{vmatrix} 2 & 5 & 4 & 7 \\ 0 & 1 & 3 & 8 \\ 9 & 4 & 7 & 8 \\ 7 & 8 & 4 & \sqrt{3} \end{vmatrix} \right)^{3-1} \begin{vmatrix} 2 & 5 & 4 & 7 & 6 & 1 & 2 \\ 0 & 1 & 3 & 8 & 8 & 1 & 5 \\ 9 & 4 & 7 & 8 & 9 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 2 & 0 & 8 \\ 11 & 2 & 5 & 4 & 5 & \frac{1}{2} & 5 \\ 5 & 7 & 8 & 6 & 1 & 0 & 5 \\ 9 & 2 & 3 & 5 & 8 & 5 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} \begin{vmatrix} 2 & 5 & 4 & 7 & 6 \\ 0 & 1 & 3 & 8 & 8 \\ 9 & 4 & 7 & 8 & 9 \\ 7 & 8 & 4 & \sqrt{3} & 2 \\ 11 & 2 & 5 & 4 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 7 & 1 \\ 0 & 1 & 3 & 8 & 1 \\ 9 & 4 & 7 & 8 & 8 \\ 7 & 8 & 4 & \sqrt{3} & 0 \\ 11 & 2 & 5 & 4 & \frac{1}{2} \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 7 & 2 \\ 0 & 1 & 3 & 8 & 5 \\ 9 & 4 & 7 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 8 \\ 11 & 2 & 5 & 4 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 & 4 & 7 & 6 \\ 0 & 1 & 3 & 8 & 8 \\ 9 & 4 & 7 & 8 & 9 \\ 7 & 8 & 4 & \sqrt{3} & 2 \\ 5 & 7 & 8 & 6 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 7 & 1 \\ 0 & 1 & 3 & 8 & 1 \\ 9 & 4 & 7 & 8 & 8 \\ 7 & 8 & 4 & \sqrt{3} & 0 \\ 5 & 7 & 8 & 6 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 7 & 2 \\ 0 & 1 & 3 & 8 & 5 \\ 9 & 4 & 7 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 8 \\ 5 & 7 & 8 & 6 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 & 4 & 7 & 6 \\ 0 & 1 & 3 & 8 & 8 \\ 9 & 4 & 7 & 8 & 9 \\ 7 & 8 & 4 & \sqrt{3} & 2 \\ 9 & 2 & 3 & 5 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 7 & 1 \\ 0 & 1 & 3 & 8 & 1 \\ 9 & 4 & 7 & 8 & 8 \\ 7 & 8 & 4 & \sqrt{3} & 0 \\ 9 & 2 & 3 & 5 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 5 & 4 & 7 & 2 \\ 0 & 1 & 3 & 8 & 5 \\ 9 & 4 & 7 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 8 \\ 9 & 2 & 3 & 5 & 3 \end{vmatrix} \end{vmatrix} \text{,finally:}$$

$$\left(89\sqrt{3}-23\right)^2 \begin{vmatrix} 2 & 5 & 4 & 7 & 6 & 1 & 2 \\ 0 & 1 & 3 & 8 & 8 & 1 & 5 \\ 9 & 4 & 7 & 8 & 9 & 8 & 6 \\ 7 & 8 & 4 & \sqrt{3} & 2 & 0 & 8 \\ 11 & 2 & 5 & 4 & 5 & \frac{1}{2} & 5 \\ 5 & 7 & 8 & 6 & 1 & 0 & 5 \\ 9 & 2 & 3 & 5 & 8 & 5 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} 194\sqrt{3}-848 & \frac{2109}{2}-\frac{1587}{2}\sqrt{3} & 163\sqrt{3}-3091 \\ 2406-1100\sqrt{3} & -371\sqrt{3}-2163 & 8146-148\sqrt{3} \\ 815\sqrt{3}-2158 & 2119-177\sqrt{3} & 265\sqrt{3}-7510 \end{vmatrix}.$$

2 Notations

For this purpose we need some notations.

Notation 1: The $(n-k)\times(n-l)$ matrix obtained from A by removing the $i_1^{th}, i_2^{th}...i_k^{th}$ rows and the j_1^{th}, j_2^{th}; $...j_l^{th}$ columns is denoted by $(a_{i,j})_{\substack{i\neq i_1,i_2,...i_k \\ j\neq j_1,j_2,...j_l}}$.

Notation 2: We denote by $\det\limits_{\alpha\leq i,j\leq\beta}\left[(a_{i,j})_{\substack{i\neq i_1,i_2,...i_k \\ j\neq j_1,j_2,...j_k}}\right]$, $k\leq\beta-\alpha$ to the determinant of the $(\beta-\alpha-k+1)$ square matrix obtained from A by removing the$(\alpha-1)$ first rows and columns, by removing the $(n-\beta)$ last rows and columns and by removing $i_1^{th}, i_2^{th}...i_k^{th}$ rows and the j_1^{th}, j_2^{th}; $...j_k^{th}$ columns.

Notation 3: We denote by $S_m(k,l)$ to the $(n-(m-1))$ square matrix obtained from A by removing the $(m-1)$ last rows and columns, and then by replacing the last row with the $(n-m+k)$ row and replacing the last column with the $(n-m+l)$ column, i.e:

$$S_m(k,l)=\overbrace{\begin{pmatrix} & & & & a_{1,n-m+l} \\ & & & & a_{2,n-m+l} \\ & (a_{i,j})_{1\le i,j\le n-m} & & & \vdots \\ & & & & a_{n-m,n-m+l} \\ a_{n-m+k,1} & a_{n-m+k,2} & \dots & a_{n-m+k,n-m} & a_{n-m+k,n-m+l} \end{pmatrix}}^{n-(m-1)}, \quad (3)$$

also it can be written as a block matrix :

$$S_m(k,l)=\overbrace{\begin{pmatrix} & a_{1,n-m+l} \\ & a_{2,n-m+l} \\ D(k) & \vdots \\ & a_{n-m,n-m+l} \\ & a_{n-m+k,n-m+l} \end{pmatrix}}^{n-(m-1)}$$

when

$$D(k)=\overbrace{\left.\begin{pmatrix} & (a_{i,j})_{1\le i,j\le n-m} & & \\ a_{n-m+k,1} & a_{n-m+k,2} & \dots & a_{n-m+k,n-m} \end{pmatrix}\right\}}^{n-m} n-(m-1).$$

Notation 4: We denote by $H_m(k,l)$ to the $(n-(m-1))$ square matrix obtained from A by deleting the $(m-1)$ rows which are: $(i_1,\dots i_{k-1},i_{k+1},\dots i_m)$ and by deleting $(m-1)$ columns which are: $(j_1,j_2,\dots i_{l-1},i_{l+1},\dots j_m)$, i.e:

$$H_m(k,l)=\left((a_{i,j})_{\substack{i\ne i_1,\dots i_{k-1},i_{k+1},\dots i_m \\ j\ne j_1,j_2,\dots i_{l-1},i_{l+1},\dots j_m}}\right).$$

3 Main Results

We need a lemma.

Lemma 1 *Let the n square matrix $A=(a_{i,j})_{1\le i,j\le n}$.*
$\forall m\in\mathbb{N}$ where $2\le m<n$,
If $\det\left[(a_{i,j})_{1\le i,j\le n-m}\right]=0$, thus we get the following formula:

$$\det_{1\le k,l\le m}\left[\det\left[S_m(k,l)\right]\right]=0. \quad (4)$$

Proof. To prove formula (4), we let $\det\left((a_{i,j})_{1\leq i,j\leq n-m}\right)=0$. But $\det\left((a_{i,j})_{1\leq i,j\leq n-m}\right)=0$ it means the rows of the matrix $(a_{i,j})_{1\leq i,j\leq n-m}$ are dependent linearly. Consequently, $\exists\,(\lambda_i)_{i=1}^{n-m}$, $\exists r\in\{1,2,..,n-m\}$ where:
$\det\left[S_m(k,l)\right]=$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1k} & \cdots & a_{1n-m} & a_{1,n-m+l} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} & \cdots & a_{2n-m} & a_{2,n-m+l} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3k} & \cdots & a_{3n-m} & a_{3,n-m+l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \sum_{i=1}^{i=n-m}\lambda_i a_{i,n-m+l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,k} & \cdots & a_{n-2,n-m} & a_{n-m,n-m+l} \\ a_{n-m+k,1} & a_{n-m+k,2} & a_{n-m+k,3} & \cdots & a_{n-m+k,k} & \cdots & a_{n-m+k,,n-m} & a_{n-m+k,n-m+l} \end{vmatrix}.$$

And simply, we get:

$$\det\left[S_m(k,l)\right]=(-1)^{n-m-r+1}\left(\sum_{i=1}^{i=n-m}\lambda_i a_{i,n-m+l}\right)\det\left(D(k)\right)_{i\neq r},$$

which we can simplify it as follows:

$$\begin{aligned} \det_{1\leq k,l\leq m}\left[\det\left[S_m(k,l)\right]\right] &= \det_{1\leq k,l\leq m}\left[(-1)^{n-m-r+1}\left(\sum_{i=1}^{i=n-m}\lambda_i a_{i,n-m+l}\right)\det\left(D(k)\right)_{i\neq r}\right] \\ &= \left[(-1)^{n-m-r+1}\right]^m\times \\ &\quad \det_{1\leq k,l\leq m}\left[\left(\sum_{i=1}^{i=n-m}\lambda_i a_{i,n-m+l}\right)\det\left(D(k)\right)_{i\neq r}\right]. \\ &= \left[(-1)^{n-m-r+1}\right]^m\times\prod_{l=1}^{m}\left(\sum_{i=1}^{i=n-m}\lambda_i a_{i,n-m+l}\right)\times \\ &\quad \prod_{k=1}^{m}\det\left(D(k)\right)_{i\neq r}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}. \end{aligned}$$

But as it is known:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}=0.$$

Thus formula (4) is realised. ■

One of the main results of the paper is the following:

Theorem 2 (Sylvester's Identity) *Let the n square matrix $A=(a_{i,j})_{1\leq i,j\leq n}$ $\forall m\in\mathbb{N}$ where $2\leq m<n$, thus the following formula is realised*

$$\det\left[(a_{i,j})_{1\leq i,j\leq n}\right]\left\{\det\left[(a_{i,j})_{1\leq i,j\leq n-m}\right]\right\}^{m-1}=\det_{1\leq k,l\leq m}\left[\det\left[S_m(k,l)\right]\right]. \quad (5)$$

We notice that this formula enables us condense the determinant of n square matrix to the determinant of m square matrix. The elements of m square matrix are the determinants of $(n-(m-1))$ square matrix.

Proof. To prove formula (5) there are two cases:

The first case: when $\det\left[(a_{i,j})_{1\leq i,j\leq n-m}\right]=0$, the proof of formula (5) is the same proof of lemma1.

The second case: when $\det\left[(a_{i,j})_{1\leq i,j\leq n-m}\right]\neq 0$ we prove formula (5) inductively:

For $m=2$, formula (5) is the same as formula (1) where $k=n-1$, $l=n$, and it has been proved (see S.Kouachi, S.Abdelmalek and B.Rebai [4]).

For $m>2$ We suppose the formula (5) is correct for $(m-1)$:

$$\det\left[(a_{i,j})_{1\leq i,j\leq n}\right]\left\{\det\left[(a_{i,j})_{1\leq i,j\leq n-(m-1)}\right]\right\}^{m-2}=\det_{1\leq k,l\leq m-1}\left[\det\left[S_{m-1}(k,l)\right]\right] \quad (6)$$

and we prove it for m.

We apply formula (1) where $k=n-1$, $l=n$ on matrix $S_{m-1}(k,l)$, we get:

$$\det\left[S_{m-1}(k,l)\right]\det\left[(a_{i,j})_{1\leq i,j\leq n-m}\right]= \\ \det\begin{bmatrix}\det\left[(a_{i,j})_{1\leq i,j\leq n-m+1}\right] & \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+2\\ j\neq n-m+1}}\right]\\ \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+1\\ j\neq n-m+2}}\right] & \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+1\\ j\neq n-m+1}}\right]\end{bmatrix}. \quad (7)$$

By using formula (7) , formula (6) will be as follows:

$$\det\left[(a_{i,j})_{1\leq i,j\leq n}\right]\left\{\det\left[(a_{i,j})_{1\leq i,j\leq n-(m-1)}\right]\right\}^{m-2}\left\{\det\left[(a_{i,j})_{1\leq i,j\leq n-m}\right]\right\}^{m-1} \\ =\det_{1\leq k,l\leq m-1}\left[\det\begin{bmatrix}\det\left[(a_{i,j})_{1\leq i,j\leq n-m+1}\right] & \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+2\\ j\neq n-m+1}}\right]\\ \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+1\\ j\neq n-m+2}}\right] & \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+1\\ j\neq n-m+1}}\right]\end{bmatrix}\right]. \quad (8)$$

We apply formula (2) on $\det_{1\leq i,j\leq m}[\det S_m(k,l)]$ to get:

$$\left(\det\left[(a_{i,j})_{1\leq i,j\leq n-m+1}\right]\right)^{m-2}\det_{1\leq i,j\leq m}[\det S_m(k,l)] =$$
$$\det_{1\leq k,l\leq m-1}\left[\det\left[\begin{array}{cc}\det\left[(a_{i,j})_{1\leq i,j\leq n-m+1}\right] & \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+2\\ j\neq n-m+1}}\right]\\ \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+1\\ j\neq n-m+2}}\right] & \det\left[S_{m-1}(k,l)_{\substack{i\neq n-m+1\\ j\neq n-m+1}}\right]\end{array}\right]\right]. \tag{9}$$

By using formula (9) , formula (8) will be as follows:

$$\det\left[(a_{i,j})_{1\leq i,j\leq n}\right]\left\{\det\left[(a_{i,j})_{1\leq i,j\leq n-(m-1)}\right]\right\}^{m-2}\left\{\det\left[(a_{i,j})_{1\leq i,j\leq n-m}\right]\right\}^{m-1}$$
$$=\left(\det\left[(a_{i,j})_{1\leq i,j\leq n-m+1}\right]\right)^{m-2}\det_{1\leq i,j\leq m}[\det S_m(k,l)].$$

According to the principle of recurrence $\left(\det\left[(a_{i,j})_{1\leq i,j\leq n-m+1}\right]\neq 0\right)$, we get formula (5) . ■

We can generalize theorem 2 by the following theorem:

Theorem 3 *Let the n square matrix $A=(a_{i,j})_{1\leq i,j\leq n}$ $\forall m\in\mathbb{N}$ where $2\leq m<n$, we can generalize formula (5) as follows:*

$$\det\left[(a_{i,j})_{1\leq i,j\leq n}\right]\left\{\det\left[(a_{i,j})_{\substack{i\neq i_1,i_2,\ldots i_m\\ j\neq j_1,j_2,\ldots j_m}}\right]\right\}^{m-1}=\det_{1\leq k,l\leq m}[\det[H_m(k,l)]]. \tag{10}$$

Proof. We move row i_m to the position of row n ; we replace row i_m and row (i_m+1) by each other. Then, the new row (i_m+1) and row (i_m+2) by each other and so on till row i_m in matrix A will be the last row. Thus, we have done $(n-i_m)$ replacings.

In the same way, to move row $i_{(m-k)}$ to the position of row $(n-k)$, we need $\left((n-k)-i_{(m-k)}\right)$ replacings where $k=0,..,m-1$.

We move column j_m to the position of column n ; we replace column j_m and column (j_m+1) by each other. Then, the new column (j_m+1) and column (j_m+2) by each other and so on till column j_m in matrix A will be the last column. Thus, we have done $(n-j_m)$ replacings.

In the same way, to move column $j_{(m-l)}$ to the position of column $(n-l)$, we need $\left((n-l)-j_{(m-l)}\right)$ replacings where $l=0,..,m-1$.We get a new matrix B that realises:

$$\begin{aligned}\det A &= (-1)^{\sum_{k=0}^{m-1}((n-k)-i_{m-k})+\sum_{k=0}^{m-1}((n-l)-j_{m-l})}\det B\\ &= (-1)^{\sum_{k=1}^{m}(n-m+k-i_k)+\sum_{l=1}^{m}(n-m+l-j_l)}\det B.\end{aligned} \tag{11}$$

We apply formula (5) on matrix B, we get:

$$\det[B]\left\{\det\left[(a_{i,j})_{\substack{i\neq i_1,i_2,\dots i_m\\ j\neq j_1,j_2,\dots j_m}}\right]\right\}^{m-1}=\det_{1\le k,l\le m}\left[\det\left[S_m(k,l)\right]\right], \tag{12}$$

when

$$S_m(k,l)=\overbrace{\begin{pmatrix} & & & & a_{(1,j_l)}\\ & & & & a_{(2,j_l)}\\ & (a_{i,j})_{\substack{i\neq i_1,i_2,\dots i_m\\ j\neq j_1,j_2,\dots j_m}} & & & \vdots\\ & & & & a_{(n-m,j_l)}\\ a_{(i_k,1)} & a_{(i_k,2)} & \dots & a_{(i_k,n-m)} & a_{(i_k,j_l)}\end{pmatrix}}^{n-(m-1)}.$$

By using the determinant properties, we get:

$$S_m(k,l)=(-1)^{((n-m+k)-i_k)+((n-m+l)-j_l)}\,H_m(k,l). \tag{13}$$

By using formula (13), formula (12) will be as follows:

$$\begin{aligned}&\det B\left\{\det\left[(a_{i,j})_{\substack{i\neq i_1,i_2,\dots i_m\\ j\neq j_1,j_2,\dots j_m}}\right]\right\}^{m-1}\\ &=\det_{1\le k,l\le m}\left[\det\left[S_m(k,l)\right]\right]\\ &=\det_{1\le k,l\le m}\left[(-1)^{((n-m+k)-i_k)+((n-m+l)-j_l)}\det\left[H_m(k,l)\right]\right]\\ &=(-1)^{\sum_{k=1}^{m}(n-m+k-i_k)+\sum_{l=1}^{m}(n-m+l-j_l)}\det_{1\le k,l\le m}\left[\det\left[H_m(k,l)\right]\right]\end{aligned}$$

This equality can be written as follows:

$$\left[(-1)^{\sum_{k=1}^{m}(n-m+k-i_k)+\sum_{l=1}^{m}(n-m+l-j_l)}\det B\right]\left\{\det\left[(a_{i,j})_{\substack{i\neq i_1,i_2,\dots i_m\\ j\neq j_1,j_2,\dots j_m}}\right]\right\}^{m-1}=\det_{1\le k,l\le m}\left[\det\left[H_m(k,l)\right]\right]$$

By using formula (11), we get formula (10). This finishes the proof. ■

Remark 1 *By using last theorem, we can write Dodgson's Algorithm in amore general way than formula (1). It will be as follows:*

$$\det\left[(a_{i,j})_{1\le i,j\le n}\right]\det\left[(a_{i,j})_{\substack{i\neq i_1,i_2\\ j\neq l_1,l_2}}\right]=$$
$$\det\left[(a_{i,j})_{\substack{i\neq i_1\\ j\neq j_1}}\right]\det\left[(a_{i,j})_{\substack{i\neq i_2\\ j\neq j_2}}\right]-\det\left[(a_{i,j})_{\substack{i\neq i_1\\ j\neq j_2}}\right]\det\left[(a_{i,j})_{\substack{i\neq i_2\\ j\neq j_1}}\right],$$

for all $1\le i_1<i_2,\ j_1<j_2\le n$.

References

[1] S. Abdelmalek, and S. Kouachi, Condensation of determinants, arXiv:0712.0822.

[2] A. G. Akritas, E. K. Akritas, and G. I. Malaschonok, Various proofs of Sylvester's (determinant) identity, Math. and Computers in Simulation 42 (1996) 585-593.

[3] C.L. Dodgson, Condensation of Determinants, Proceedings of the Royal Society of London, 15 (1866), 150-155.

[4] S. Kouachi, S. Abdelmalek, and B. Rebai, A Mathematical Proof of Dodgson's Algorithm, arXiv:0712.0362.

[5] T. Muir, The Theory of determinants in The Historical Order of Development, vol. II. London (1911).

Received: December 14, 2007