

# WCDMA Uplink System Capacity: A Singular Perturbation Approach<sup>1</sup>

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## Abstract

We consider a WCDMA system with two types of calls: real time (RT) calls that have dedicated resources, and data on real time (BE) calls (i.e. best effort) that share system capacity. We consider reservation of some capacity resources for the BE traffic as well as any capacity left over from RT calls. Our analysis approach is based on modeling of the system as a two dimensional Markov chain, where the first correspond to the number of RT calls and the second to the number of BE calls in the system. In order obtain the steady state distribution of this system, we use a singular perturbation solution approach for approximating the steady state. Our approach gives a good approximation and relatively faster computation in comparison with our exact methods like spectral analysis [17]. Based on this analysis, we derive performance evaluation results regarding blocking of RT calls and sojourn time of BE calls, under different traffic characteristics. Finally, we extend our result to cover BE admission control. Here, we find the maximum BE arrival rate such that the sojourn time of BE calls not exceed a threshold.

**Keywords:** Singular W-CDMA, singular perturbation, Lyapunov, quasi-birth-death process.

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## 1 Introduction

The Universal Mobile Telecommunication System (UMTS) operates with Wideband Code Division Multiple Access (WCDMA) over the air interface. The advantage of 3G (third generation of mobile networks) resides in the fact that they offer to users a large possibility of services. These services are related to real time and best effort applications like transferring files, emailing, etc. Each service has a demand of quality of service. The variations caused by the diversity of classes can affect the WCDMA capacity.

There are many existing works on the capacity analysis for wireless networks. Several research axes on W-CDMA capacity has been considered. In [25], the authors present a method to calculate the WCDMA reserve link Erlang based on the Lost Hel (LCH) model as described in [24]. [10] This algorithm calculate the occupancy and capacity UMTS/WCDMA systems based on a system outage condition. In this research, the authors derive a closed form expression of Erlang capacity for a single type of traffic. The capacity of an uplink with two classes is considered in [15] in which the real-time (RT) traffic is transmitted all the time, the non real time mobiles (BE) are time-shared . In [1], the author considers best-effort (BE) and real-time (RT) applications. He study the influence of the value of a fixed (not-adaptive) bandwidth per BE calls on the Erlang capacity of the system (that includes also RT calls), taking into account that a lower bandwidth implies longer call durations. In [13], the authors extend the notion of capacity in [1] to other QoS. The delay aware capacity, suitable in particular for the BE traffic, is defined as the arrival rate of BE calls that the system can handle such that their expected delay is bounded. In ad hoc networks, there many existing works on the capacity analysis [9, 11, 18, 21]. The performance of ad hoc network based on 802.11 is studied in [7, 23, 8, 16, 12].

In this paper, we consider a uplink WCDMA system with two types of calls : real time (RT) calls that have a dedicate resources , and BE traffic without any QoS. The first objective of this paper is to compute the distribution of number of RT connections and the number of BE connections in the steady-state. By using the singular perturbation, we simplify the computation of this distribution. After this, we provide a closed form of that distribution. We shall apply this method to the case when the transition of the number of RT calls occur much more frequently than those of the number of BE calls. This approach allows us to compute different performances measures : blocking probability, expected delay and throughput.

Our analysis approach is based on modeling of the system as a two dimensional Markov chain, where the first corresponds to the number of (RT) calls in the system and the second to the number of (BE) calls in the system. In order to obtain the steady state distribution of this system we make use of singular

perturbation methods of analysis of quasi birth and death (QBD) processes [22, 6, 5].

The objective of this paper is to study a singular perturbation solution approach for approximating the steady state solution. This approach allows us to obtain a simple approximation for the case that the number of RT calls evolves much faster than the number of BE calls. The singular perturbation approach allows us to represent the steady state probabilities as a Taylor series. The first term in the series already gives a good approximation.

Singular perturbation is applied in modeling problems as well as control and optimization in the literature. We find for example the singular perturbation chains under weak and strong interactions, between groups and within group members respectively [22, 6, 5, 3, 4]. This approach is a good method to obtain simple approximation of steady-state probabilities of Markov chains with two time scales : short-time scale and long-time scale. The transitions between group of states are less frequent than within each group.

The structure of this rapport is as follows. Next section 2 introduces the model and the problem formulation. Section 3, we use singular perturbation approach to derive a closed of steady-states probabilities and compute the performance of RT and BE traffic. In section 4, we provide numerical examples and we compare our approach with exact solution which obtained by spectral analysis approach [17]. Finally, section 5 concludes the paper.

## 2 Problem formulation

Let  $S = \{1, \dots, C\}$  be the set of multi-service classes in a uplink of a WCDMA system with multi-sectors. Let  $M_i$  be the number of mobiles of class  $i$  where  $M_1, \dots, M_C$  mobiles are in a sector. We define the received power from mobile of class  $i$ , at the base station in the sector by  $(P_i)$ . This power is the same for all mobiles of class  $i$  and the following signal to interference ratio (SIR) expression should be satisfied in order to have uplink communication [19] :

$$\frac{P_i}{N + I_{own} + I_{other} - P_i} \geq \left(\frac{E_i R_i}{N_0 W}\right)_i$$

where  $i = 1, 2, \dots, C$ ,  $N$  is the background noise density,  $E_i$  is the energy per transmitted bit of type  $i$ ,  $R_i$  is the transmission rate of class  $i$  service,  $W$  is the spread-spectrum bandwidth,  $I_{own}$  is the total power received from mobiles belong in same sector, and  $I_{other}$  denote the total power received from mobiles in other sectors.

By definition, the intra-cell interferences in a sector is given by

$$I_{own} = \sum_{i=1}^C M_i P_i$$

then  $I_{other}$  can be obtained from  $I_{other} = gI_{own}$  (see [19]), where  $g$  is the constant of interference given from measurement. In the WCDMA other sources of degradation i.e. shadow fading, etc are exist. In order to take into account shadow fading, the authors in [13] introduced a new constant ( $\Gamma$ ) independent of class type. So the signal to interference ration (SIR) must be larger than  $(\frac{E_i R_i}{N_0 W} \Gamma)_i$ . Then for a better satisfaction of calls of the class  $i$  without degradation in QoS, the minimal received power ( $P_i$ ) must satisfy the following [13]:

$$P_i = \frac{N \Delta_i}{1 - (1 + g) \sum_{i=1}^C M_i \Delta_i}$$

where  $\Delta_i = \frac{\alpha_i}{1 + \alpha_i}$ ,  $\alpha_i = (\frac{E_i R_i}{N_0 W} \Gamma)_i$ , and the load rate is defined as:  $\theta = \sum_{j=1}^C M_j \Delta_j$ .  $(1 + g) \Delta_i$  in above expression is the capacity required by a call of type  $i \in S$ .

In this paper we consider two types of calls: real time (RT) calls that have dedicated resources and data Best Effort (BE) calls that share system capacity. The upper bound for the capacity of uplink of the CDMA system (i.e.  $(\theta < \theta_{\epsilon'})$ ) [2, 13] with single sector is given by:  $\theta_{\epsilon'} = 1 - \epsilon'$  where  $\epsilon' > 0$ . Let  $L_{RT}$  (resp.  $L_{BE}$ ) be the the capacity used by RT (resp. BE) calls with  $L_{RT} + L_{BE} = \theta_{\epsilon'}$ .

We suppose UMTS uses the AMR<sup>2</sup> codec. This codec offer eight different transmission rates of voice that vary between 4.75 kbps tp 12.2 kbps, and that can dynamically changed every 20 msec. Hence, we assume that the set available transmission rates for **RT** traffic has the form  $[R_{min}^{RT}, R_{max}^{RT}]$ , where  $R_{min}^{RT}$  is the minimum rate and  $R_{max}^{RT}$  is the maximum rate. The normalized bandwidth corresponding to both rates are  $\Delta_{min}^{RT}$  and  $\Delta_{max}^{RT}$  respectively. Then we have

$$\Delta_{max}^{RT} = \frac{E_{RT}/N_0}{W/R_{max}^{RT} + E_{RT}/N_0}$$

and

$$\Delta_{min}^{RT} = \frac{E_{RT}/N_0}{W/R_{min}^{RT} + E_{RT}/N_0}$$

where  $E_{RT}/N_0$  is the SIR (signal-to-interference ratio) required for a RT calls and  $W$  is the WCDMA modulation bandwidth.

The number of RT calls respectively corresponding to  $R_{max}^{RT}$  and  $R_{min}^{RT}$  is given by  $M_{RT} = \lfloor L_{RT}/\Delta_{min}^{RT} \rfloor$  and  $N_{RT} = \lfloor L_{RT}/\Delta_{max}^{RT} \rfloor$  where  $L_{RT}$  is the capacity reserved of RT calls. Thus the bandwidth required by each **RT** call is

$$\Delta(i) = \begin{cases} \Delta_{max}^{RT}, & \text{if } i \leq N_{RT}; \\ L_{RT}/i, & \text{for } N_{RT} < i \leq M_{RT}. \end{cases}$$

Recall that the process  $X(t)$  is a birth and death process, with birth rate  $\lambda_{RT}$  and death rate  $\mu_{RT}$ . The probability of accepting a new RT call is given

<sup>2</sup>Adaptation Multi-Rate (AMR) codec used by UMTS that offers eight transmission rate varying between 4.75 kbps and 12.2 kbps.

by  $P[X = i] = P[\lim_{t \rightarrow \infty} X(t) = i] = \frac{\rho_{RT}^i / i!}{q}$ , where  $q = \sum_{i=0}^{M_{RT}} \frac{\rho_{RT}^i}{i!}$  and  $\rho_{RT} = \frac{\lambda_{RT}}{\mu_{RT}}$ . Therefore, the blocking probability of a new RT calls is given by  $P_B^{(RT)} = P[X = M_{RT}]$  as follows :

$$P_B^{(RT)} = \frac{\rho_{RT}^{M_{RT}} / M_{RT}!}{\sum_{i=0}^{M_{RT}} \frac{\rho_{RT}^i}{i!}}$$

We consider that BE calls make use of the reserved system capacity, as well as any capacity left over from RT calls. Thus the BE and RT calls share a common portion of capacity between them. Thus the available capacity of BE calls is a function of the number of RT calls in the systems:

$$C(i) = \begin{cases} \theta_\epsilon - i\Delta_{max}^{RT}, & \text{if } i \leq N_{RT}; \\ L_{BE}, & \text{for } N_{RT} < i \leq M_{RT}. \end{cases}$$

where  $E_{BE}/N_0$  is the required SIR of **BE** of the call. However, in order to guarantee a *QoS* for **BE** calls, a maximal number of **BE** calls ( $M_{BE}$ ) should be computed. The departure rate of BE calls is depended on the current number of RT calls given by  $v(i) = \mu_{BE}R_{BE}(i)$ , where  $R_{BE}(i) = \frac{C(i)W}{(1-C(i))E_{BE}/N_0}$  and  $E_{BE}$  is the energy per bit transmitted for BE calls. We define the total transmission rate of RT calls , as follow:  $R_T = \lfloor \frac{\theta_\epsilon}{\Delta_{min}^{RT}} \rfloor R^m$ , the normalized load for RT calls by  $\bar{\rho}_{RT} = \frac{\lambda_{RT}}{\mu_{RT}} \frac{R^r}{R_T}$  and by  $\bar{\rho}_{BE} = \rho_{BE} \times \frac{1}{R_T}$  normalized load for BE calls, where  $\rho_{BE} = \frac{\lambda_{BE}}{\mu_{BE}}$ .

We assume that RT and BE calls arrive according to independent Poisson processes with rates  $\lambda_{RT}$  and  $\lambda_{BE}$ , respectively. The duration of an RT call is exponentially distributed with parameter  $\mu_{RT}$ . The size of an BE file is exponentially distributed with parameter  $\mu_{BE}$ . Interval times, RT call duration and BE file sizes are all independent.

Let  $X(t)$  (resp.  $Y(t)$ ) be the number of RT (resp. BE) calls in the system at time  $t$ . The Markov chain  $(X(t), Y(t))$  is a birth and death process of two dimension, with birth rates  $\lambda_{RT}$  and  $\lambda_{BE}$ , and death rates  $\mu_{RT}$  and  $\mu_{BE}$ .

The performances of the uplink WCDMA system is usually investigated by studying the blocking probability, sojourn time and the throughput. These parameters are complex to compute. We need to obtain steady state distribution of the number of RT calls and BE calls in order to investigate these performance metrics. In paper [17], we obtain a closed form of the steady state, but remains difficult to compute it. In this paper we introduce a simple method that offers simpler computation under some specific conditions.

The conditions which makes us able to use singular perturbation is the following: In a typical WCDMA system BE services (e.g. chat service) remain for a longer time, but the RT services (e.g. voice call) in generally takes less time. We thus represent the transition rate as follow :

$$\lambda_{BE}^\epsilon = \epsilon\lambda_{BE}, \quad \mu_{BE}^\epsilon = \epsilon\mu_{BE}$$

where  $\epsilon$  is small strictly positive parameter. Hence the transition matrices of the Markov chain  $(X(t), Y(t))$  is given by:

$$Q(\epsilon) = Q_0 + \epsilon Q_1 \tag{1}$$

where  $Q(\epsilon)$  is a generator correspond to perturbed Markov chain,  $Q_0$  is a generator unperturbed correspond to strong interactions, and  $\epsilon Q_1$  is perturbed term for weak interactions; i.e.,

$$Q_0 = \begin{bmatrix} Q_{RT} & 0 & 0 & \dots \\ 0 & Q_{RT} & 0 & \dots \\ 0 & 0 & Q_{RT} & \dots \\ 0 & 0 & \ddots & \ddots \end{bmatrix}$$

and

$$Q_1 = \begin{bmatrix} -A_0 & A_0 & 0 & 0 & \dots \\ A_2 & -A_2 - A_0 & A_0 & 0 & \dots \\ 0 & A_2 & -A_2 - A_0 & A_0 & \dots \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

where the matrices  $A_0$ ,  $A_2$  and  $Q_{RT}$  are square matrices of size  $(M_{RT} + 1)$ . The matrix  $A_0$  corresponds to a BE calls arrival, i.e.,  $A_0 = \text{diag}(\lambda_{BE})$ . The matrix  $A_2$  corresponds to departure of a BE calls, i.e.,  $A_2 = \text{diag}(v(i))$ ,  $i = 0, 1, 2, \dots, M_{RT}$  where  $i$  is the number of RT calls. The matrix  $Q_{RT}$  corresponds to the arrival and departure process of RT calls. This matrix is tri-diagonal as follows:

$$\begin{aligned} Q_{RT}[i, i - 1] &= i\mu_{RT}, \\ Q_{RT}[i, i] &= -(\lambda_{RT} + i\mu_{RT}), i = 0, \dots, M_{RT} - 1, \\ Q_{RT}[M_{RT}, M_{RT}] &= -M_{RT}\mu_{RT}, \\ Q_{RT}[i, i + 1] &= \lambda_{RT}. \end{aligned}$$

Of course,  $Q_{RT}$  it's the generator of queuing model  $M/M/M_{RT}/M_{RT}$  with arrived rate  $\lambda_{RT}$  and rate departure  $i\mu_{RT}$  for  $i = 0, \dots, M_{RT}$ .

Let  $\pi^{(\epsilon)}(i, j) = \lim_{t \rightarrow \infty} P(X(t) = i, Y(t) = j) = P(X = i, Y = j)$  be the probability distribution of the number RT and BE in the system. Since  $Q(\epsilon)$  is irreducible ( $\epsilon > 0$ ), then the steady state solution is unique and satisfies the following system:

$$\pi^{(\epsilon)}Q(\epsilon) = 0 \tag{2}$$

$$\pi^{(\epsilon)}\mathbf{1} = 1 \tag{3}$$

where  $\mathbf{1}$  is a vector of ones with the same dimension as  $\pi^{(\epsilon)}$ . The states space of Markov chain  $\mathbf{E} = \{(i, j), 0 \leq i \leq M_{RT}, j \geq 0\}$  is partitioned in subsets  $j$  such as  $\mathbf{B} = \{l(0), l(1), l(2), \dots\}$ . where  $l(j) = \{(0, j), \dots, (M_{RT}, j)\}$  for  $j \geq 0$  and  $l(j)$  is a group of  $RT$  calls correspond to a  $j$   $BE$  calls. The matrix  $Q_0$  contains several ergodic classes  $j$  where each class has a generator  $Q_{RT}$  as block diagonal in the matrix  $Q_0$ . This distribution allows us to compute the average number of  $BE$  calls, and expected delay of sojourn time of  $BE$  calls as follows:

$$E_\epsilon(Y) = \sum_{j=0}^{\infty} j P_\epsilon^{(n)}(Y = j),$$

and

$$T_\epsilon^{BE} = \frac{E_\epsilon(Y)}{\lambda_{BE}^\epsilon}$$

where  $P_\epsilon^{(n)}(Y = j) = \sum_{i=0}^{M_{RT}} \pi^{(\epsilon)}(i, j)$ , and  $n$  is order of approximate in Taylor series, see lemma 3.4 in the next section.

**Remark 2.1.** *Alternatively, our analysis is equally applies when transitions within the groups (i.e  $RT$  calls occurred rapidly) are much more frequent than between groups (i.e  $BE$  calls occurred slowly). In this case, the arrived rate  $\lambda_{RT}^\epsilon = \frac{1}{\epsilon} \lambda_{RT}$  and the departure rate  $\mu_{RT}^\epsilon = \frac{1}{\epsilon} \mu_{RT}$ . This formulation represent the strong interactions within groups, can be writing in multiple of  $\frac{1}{\epsilon}$ , i.e. when  $Q(\epsilon) = \frac{1}{\epsilon} Q_0 + Q_1$ , for the same matrices  $Q_0$  and  $Q_1$ . If  $\epsilon$  is very small, then the sojourn time of  $RT$  calls is very big (i.e.  $\frac{1}{\epsilon}$  goes in infinite). In this case, our system are considered as perturbed by that phenomena.*

### 3 Taylor series for steady-state distribution in the perturbation parameter $\epsilon$

Firstly we need to introduce some definitions needed in the sequel. Fixe a denumerable vector  $\delta$  with strictly positive entries. Let  $v$  be a vector defined on a subset  $\mathcal{I}$  of the nonnegative integers. Then its  $\delta$ -norm is defined as

$$\|v\|_\delta \triangleq \max_{i \in \mathcal{I}} \frac{|v_i|}{\delta_i}$$

The corresponding induced  $\delta$ -norm for any operator  $A$  on  $\mathcal{I} \times \mathcal{I}$  is given by

$$\|A\|_\delta \triangleq \max_{i \in \mathcal{I}} \frac{\sum_{j \in \mathcal{I}} |b_{ij}| \delta_j}{\delta_i}$$

The following results show that the Markov chain satisfies the following assumptions.

- Lemma 3.1.** 1. The unperturbed Markov chain has a several ergodic classes  $l(j)$ ,  $j \in I_\infty = \{0, 1, 2, \dots\}$ .
2. Each Markov chain that correspond to a ergodic class of  $Q_0$  is uniformly Lyapunov stable.
3. The aggregated Markov chain is irreducible and Lyapunov stable.
4. The generator  $Q_1$  is  $\delta$ -bounded ( $\|Q_1\|_\delta < \infty$ ).

*Proof.* 1. From the previous analysis, the unperturbed Markov chain consists of several classes  $l(j)$ ,  $j \in \mathcal{I}$  and there are no transient states. Since the values  $\lambda_{RT}$  and  $\mu_{RT}$  are positive, then the class  $l(j)$  is a finite ergodic class.

2. The operator of unperturbed Markov chain is  $P^{(0)} = r_0^{-1}Q_0 + I$  which given by

$$P^{(0)} = I + \begin{pmatrix} r_0^{-1}Q_{RT} + I & 0 & \dots \\ 0 & r_0^{-1}Q_{RT} + I & \ddots \\ 0 & \ddots & \ddots \end{pmatrix}$$

where  $I$  is a matrix square of of order  $M_{RT} + 1$ , and  $r_0^{-1} = \frac{1}{\lambda_{RT} + M_{RT}\mu_{RT}}$ . Let  $P_{RT} = 1 - \frac{\lambda_{RT}}{r_0} > 0$ . Since  $P_{RT}(i, i)1 - \frac{\lambda_{RT}}{r_0} > 0$ , then the matrix chain correspond to the ergodic classes of the unperturbed Markov chain are strongly aperiodic which implies that there exists constant  $c$  and  $\beta$  ( $c > 0, 0 < \beta < 1$ ) positive such that [see [6] fore more details]

$$\|P_{RT}^n - \mathbf{1}\tilde{q}_{l(i)}\|_\delta \leq c\beta^n, \forall n \geq 0 \text{ and } \beta < 1$$

3. Let  $\tilde{Q}_1$  is a generator of aggregated Markov chain in the space  $\mathbf{B}$  where each class  $l(j)$  is replaced by aggregated state  $j$  (in the sequel we replace  $l(j)$  by  $j$ ).

$$(\tilde{Q}_1)_{i,j} = \tilde{q}_i Q_1[i, j]\underline{\xi}, \forall (i, j) \in I_\infty \times I_\infty$$

where  $\underline{\xi} = (1, \dots, 1)^t \in \Re^{(M_{RT}+1) \times 1}$ . This aggregate matrix, represents the transition rates between groups. Each column, and likewise each row, in  $\tilde{Q}_1$  corresponds to a group in  $B$ . The matrix  $\tilde{Q}_1$  is tr-diagonal as follows:

$$\begin{aligned} \tilde{Q}_1[j, j + 1] &= \tilde{q}_j A_0 \underline{\xi} \\ \tilde{Q}_1[j, j - 1] &= \tilde{q}_j A_2 \underline{\xi} \\ \tilde{Q}_1[j, j] &= -\tilde{q}_j (A_0 + A_2) \underline{\xi}, (\forall j > 0) \\ \tilde{Q}_1[0, 0] &= -\tilde{q}_0 (A_0) \underline{\xi} \end{aligned}$$



where  $\tilde{q}_j A_n \underline{\xi} = \sum_{i=0}^{M_{RT}} \tilde{q}_j(i) (A_n)_{(i,i)}$  for  $n = 0, 2$ . The invariant distribution  $\bar{v} = [\bar{v}_0 \bar{v}_1 \dots]$  corresponded to this generator can be obtained by the following equation.

$$\bar{v} \tilde{Q}_1 = 0, \sum_{i=0}^{\infty} \bar{v}_i = 1 \tag{4}$$

The vector  $\tilde{q}_j = [\tilde{q}_j(0), \dots, \tilde{q}_j(M_{RT})]$  is unique solution of the following equation

$$\tilde{q}_j Q_{RT} = 0, \tilde{q}_j \underline{\xi} = 1 \tag{5}$$

hence, the vector  $\tilde{q}_j$  is given by

$$\tilde{q}_j(n) = \frac{(\rho_{RT})^n}{n!} \left( \sum_{i=0}^{M_{RT}} \frac{(\rho_{RT})^i}{i!} \right)^{-1}, 0 \leq n \leq M_{RT} \tag{6}$$

The generator  $\tilde{Q}_1$  becomes

$$\tilde{Q}_1 = \begin{bmatrix} -\lambda_{BE} & \lambda_{BE} & 0 & \dots \\ c & -\lambda_{BE} - c & \lambda_{BE} & \ddots \\ 0 & c & -\lambda_{BE} - c & \lambda_{BE} \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$c = \sum_{i=0}^{N_{RT}} v(i) \tilde{q}(i) + \sum_{i=N_{RT}+1}^{M_{RT}} v(i) \tilde{q}(i)$$

Let

$$\alpha(M_{RT}) = \frac{\lambda_{BE}}{\sum_{i=0}^{N_{RT}} v(i) \tilde{q}(i) + \sum_{i=N_{RT}+1}^{M_{RT}} v(i) \tilde{q}(i)}$$

The generator of aggregated Markov chain as homogenous QBD process with arrived rate is  $\lambda_{BE}$  and depart rate is  $c$ . We can show easily that the aggregated Markov chain is ergodic irreducible and strongly aperiodic, hence it's ergodic irreducible if  $\alpha(M_{RT}) < 1$ . Let  $\tilde{P} = r_{BE}^{-1} \tilde{Q}_1 + I$ , the matrix of transitions the probabilities associated to aggregated Markov chain and  $r_{BE}^{-1} = \frac{1}{\lambda_{BE} + 2v(0)}$ . We observe the state (0,0) is strongly aperiodic, because  $\tilde{P}(0,0) = 1 - \frac{\lambda_{BE}}{r_0} > \frac{v(0)}{r_0} > 0$ , then Lyapunov function  $\delta$  to hold  $\tilde{P}\delta \leq \gamma\delta + b\mathbf{1}_{(0,0)}$ , where  $b = \sqrt{c}(1 - \sqrt{\alpha(M_{RT}))}\delta_0, \delta_{l(i)} = (\sqrt{\alpha(M_{RT}))}^i$  et  $\gamma = 1 - (\lambda_{BE} - c)^2$  (see [6, 20]).

4. The norm of operator  $Q_1$  is bounded by a positive constant. By definition we have

$$\|Q_1\|_\delta \triangleq \max_{j \in I_\infty} \frac{\sum_{k \in I_\infty} \|Q_1(i, k)\| \delta_k}{\delta_j} = \max_{i \in I_\infty} \left\{ \frac{\|A_2(i, \cdot)\|_\delta (\delta_0 + \delta_1)}{\delta_0}, \frac{\|A_0\|_\delta \delta_{i-1} + \|(A_2 + A_0)\|_\delta \delta_i + \|A_0\|_\delta \delta_{i+1}}{\delta_{i-1}} \right\}$$

where

$$\|A_0\|_\delta = \lambda_{BE} \|I\|_\delta = \lambda_{BE}, \text{ and } \|A_2\|_\delta = v(0)$$

Since

$$\max(v(i)) = v(0), \text{ and } \|(A_2 + A_0)\|_\delta = \lambda_{BE} + v(0)$$

Then we have

$$\|Q_1\|_\delta = \max_{i \in I_{MRT}} \left\{ \lambda_{BE} \left(1 + \frac{\delta_1}{\delta_0}\right), \lambda_{BE} \left(1 + \frac{\delta_i}{\delta_{i-1}}\right) + v(0) \left(1 + \frac{\delta_{i+1}}{\delta_i}\right) \right\}$$

which shows that the generator  $Q_1$  is  $\delta$ -bounded □

Now, we are able to present the steady-state distribution of the perturbed Markov chain, as an analytical function of  $\epsilon$ . From lemma 3.1, the assumptions of theorem 2 in [5] are satisfied, then we have the following results

**Theorem 3.2.** *The invariant probability measure  $\pi^{(\epsilon)}$  is an analytic function of  $\epsilon$*

$$\pi^{(\epsilon)} = \pi^{(0)} + \pi^{(1)}\epsilon + \pi^{(2)}\epsilon^2 + \pi^{(3)}\epsilon^3 + \dots \quad (7)$$

where  $0 < \epsilon \leq \min(\epsilon_{max}, \frac{1}{\|U\|_\delta})$  and  $\|U\|_\delta$  is given in the proposition 3.3.

The coefficients of probabilities  $\pi^{(\epsilon)}$  are recursively calculated by the following formulas :

$$\pi^{(0)} Q_0 = 0, \quad \sum_{(i=0, j=0)}^{(M_{RT}, \infty)} \pi^{(0)}(i, j) = 1 \quad (8)$$

$$\pi^{(n)} Q_0 + \pi^{(n-1)} Q_1 = 0, \quad \sum_{(i=0, j=0)}^{(M_{RT}, \infty)} \pi^{(n)}(i, j) = 0, \quad \forall n \geq 1 \quad (9)$$

The unperturbed stationary probability is given by :  $\pi_j^{(0)} = \bar{v}_j \tilde{q}_j$ , where  $\bar{v}_j$  is solution of the equation (3) and obtained as follows :

$$\bar{v}_i = \alpha (M_{RT})^i \bar{v}_0 \text{ where } \bar{v}_0 = 1 - \alpha (M_{RT}), \forall i \in I_\infty \quad (10)$$

Let  $\xi_{(i)}$  be the right eigenvector column corresponding to the one eigenvalue of the matrix  $Q_{RT}$ . In our model this vector is given by :  $\forall j \in I_{MRT}, \forall k \in I_\infty$ , we have

$$\xi_{l(i)}(k, j) = \begin{cases} 1 & \text{if } k = i; \\ 0 & \text{else.} \end{cases}$$

Let  $\mathcal{N}$  be a matrix of eigenvalues corresponding to the zero eigenvalue of the unperturbed generator  $Q_0$ , i.e  $N = [\xi_{l(0)} \ \xi_{l(1)} \ \dots]$  and  $\mathcal{M}$  is a matrix whose rows are stationary distribution of the Markov chain  $Q_0$ , i.e  $M = \text{diag}(\tilde{q}_0, \tilde{q}_1, \dots)$ .

**Proposition 3.3.** *A solution of the equations (8)-(9) with the normalization conditions is given by the following recursive formulas:*

1.

$$\pi_{(i,j)}^{(0)} = (1 - \alpha(M_{RT}))\alpha(M_{RT})^j \frac{\frac{(\rho_{RT})^i}{i!}}{\sum_{k=0}^{M_{RT}} \frac{(\rho_{RT})^k}{k!}} \tag{11}$$

2.

$$\pi^{(n)} = \pi^{(0)}U^n, U = Q_1H[I + Q_1\Lambda\tilde{H}M] \tag{12}$$

where  $\|U\|_\delta = g_1\|H\|_\delta(1 + g_1\|\Lambda\tilde{H}M\|_\delta)$ .

The matrices  $H, \tilde{H}$  are deviation matrices of unperturbed Markov chain and aggregated Markov chain.

$$\Lambda = \begin{pmatrix} \underline{\xi} & \underline{0} & \underline{0} & \underline{0} & \cdot \\ \underline{0} & \underline{\xi} & \underline{0} & \underline{0} & \cdot \\ \underline{0} & \underline{0} & \underline{\xi} & \underline{0} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \text{ where } \underline{\xi} = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \text{ and } \underline{0} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$H = [-Q_0 + NM]^{-1} - NM = \text{diag}(H_0, \dots)$$

where  $H_0 = [q^* - Q_{RT}]^{-1} - q^*$ , and  $q^* = \underline{\xi}\tilde{q}$  ( $\tilde{q} = \tilde{q}_j$ ), then the  $\delta$ -norm of deviation matrix  $H$  is given by  $\|H\|_\delta = \|H_0\|_\delta$ .

$\tilde{H} = [-\tilde{Q}_1 + \tilde{Q}_1^*]^{-1} - \tilde{Q}_1^*$ , where  $\tilde{Q}_1^* = \xi_{\mathbf{B}}\tilde{\pi}$ ,  $\xi_{\mathbf{B}} = (1, \dots, 1) \in \mathfrak{R}^{\infty \times 1}$  and  $\tilde{H} = (\tilde{H}[i, j]), \forall i, j \in I_\infty$ .

The matrix blocs  $U_{i,j}$  of the matrix  $U = \{U_{i,j}\}_{i,j \in B}$  can be computed by using the follows formulas:

$$U_{i,j} = Q_{i,j}^1 H_j + \sum_{l \in B} Q_{i,l}^1 H_l \sum_{k \in B} Q_{l,k}^1 \tilde{H}_{k,j} \mathbf{1}q_j.$$

We observe here that our approach here is more simpler than other methods as spectral analysis (see [17]). We just need to compute the invariant distributions of each recurrent class of the unperturbed Markov chain  $Q_0$  in (5), and the invariant distributions of aggregated Markov chain  $\tilde{Q}_1$  given by (10).

**Lemma 3.4.** *The marginal distribution probability of BE calls, in the Taylor series, is given by :*

1. For first coefficient  $\pi^{(0)}$ , then

$$P_\epsilon^{(0)}(Y = j) = \pi_j^{(0)} \mathbf{1} = (1 - \alpha(M_{RT}))\alpha(M_{RT})^j$$

2. For others coefficient  $\pi^{(n)}$  ( $n \geq 1$ ), then

$$P_\epsilon^{(n)}(Y = j) = \pi_j^{(0)} \mathbf{1} + \epsilon \pi_j^{(1)} \mathbf{1} + \dots + \epsilon^n \pi_j^{(n)} \mathbf{1}$$

where  $\pi_j^{(n)} = \sum_{i \in B} \pi_i^{(0)} U_{i,j}^{(n)}$  and  $U^n = \{U_{i,j}^{(n)}\}$ .

Now, we can to compute the average marginal number  $E_\epsilon(Y)$  of BE calls, using the first coefficient Taylor (when ( $n = 0$ )), as follow :

$$E^{BE} = \frac{\alpha(M_{RT})}{1 - \alpha(M_{RT})} \tag{13}$$

The mean delay of sojourn time of BE calls, using Little formulas, as follows :

$$T^{BE} = \frac{E^{BE}}{\lambda_{BE}^\epsilon} \tag{14}$$

We can rewrite this equation as follows with threshold  $L_{BE}$  parameter:

$$T^{BE}(L_{BE}) = \frac{\frac{1}{\epsilon}}{\gamma[\sum_{i=0}^{N_{RT}} \tilde{q}(i)\beta_{\epsilon'} + \frac{L_{BE}}{1-L_{BE}} \sum_{i=N_{RT}+1}^{M_{RT}} q(i)] - \lambda_{BE}}$$

where  $\gamma = \frac{\mu_{BE}W}{E_{BE}/N_0}$  and  $\beta_{\epsilon'} = \frac{\theta_{\epsilon'} - i\Delta_{RT}^r}{1 - \theta_{\epsilon'} + i\Delta_{RT}^r}$ . We observe in the above equation, if the capacity  $L_{BE}$  of BE calls changes, then the BE sojourn time ( $T^{BE}(L_{BE})$ ) is directly influenced by this variation.

## 4 Numerical results

In this section, we test our approximation based on singular perturbation approach. We suppose that upper bound<sup>3</sup> for RT calls is 12.2 kbps and the minimum rate is 4.75 kbps. We assume  $E_{RT}/N_0 = 4.1$  dB,  $W = 3.86$  Mcps,  $E_{BE}/N_0 = 4.1$  dB [14],  $\bar{\rho}_{BE} = 0.55$ ,  $\bar{\rho}_{RT} = 0.5$ ,  $R_T = 38$  kbps and  $\theta_{\epsilon'} = 1 - 10^{-5}$ . The arrived rate of BE is  $\lambda_{BE}^\epsilon = 0.209\epsilon$  and depart rate is  $\mu_{BE}^\epsilon = 10^{-5}\epsilon$

<sup>3</sup>bounded denote the maximum rate for AMR speech service, see Holma [14].

where  $\epsilon = 10^{-3}$ . In the figure 1 we compare steady state vector of marginal probabilities obtained by two methods: spectral analysis and singular perturbation. The capacity threshold of BE is  $L_{BE} = 0.752$  and the sojourn mean time rate of RT calls is  $\mu_{RT} = 0.1$ . In this figure, we observe that the first term in Taylor series give a good approximation when the parameter of perturbation is sufficiently small ( $\epsilon = 10^{-3}$ ).

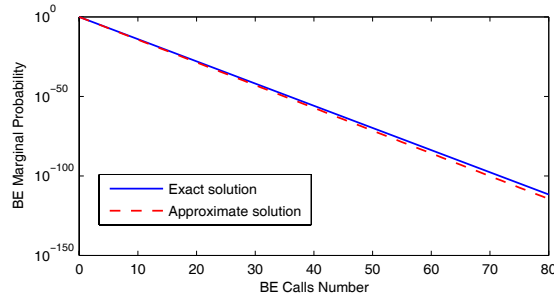


Figure 1: Marginal probability of BE calls versus versus number of the number of BE calls in the system for  $\epsilon = 10^{-3}$ ,  $\lambda_{BE}^\epsilon = 0.209\epsilon$ ,  $\mu_{BE}^\epsilon = 10^{-5}\epsilon$ ,  $\lambda_{RT} = 0.1557$ , threshold BE  $L_{BE} = 0.752$  and  $\mu_{RT} = 0.1$ .

Figures 2 and 3, shows the expected average sojourn time of BE calls in the system. We observe that the first coefficient in Taylor series gives a good approximation in the figure 2. We observe also that the error between the approximative solution (singular perturbation) and the exact solution (spectral analysis), increases when the service time of RT calls decreases. An intuitive explanation is as follow: when we use the first term of the Taylor series, we get the limiting behavior as  $\epsilon$  tends to zero, i.e, as the transition of the number of RT calls occur much faster than the transition of the number of BE calls. However, when the service time of RT calls decreases, the RT calls spend more time in the system which means that the transition between of the number of RT calls becomes less faster. However, the service provider can use this approach to evaluate in real time with fast computation the performance measures of RT calls and BE calls and decides which best configuration (BE threshold) or admission control (CAC) improves the bandwidth utilization. Indeed, these performances depends on several parameters as arrival probabilities of calls, average duration of RT call, shadow fading, ect. These parameter change dynamically and the system needs to evaluate the performance with new parameters. Hence, our approach is useful to service provider to obtain best CAC and BE threshold that maximize the bandwidth utilization and satisfy the QoS required by RT calls and BE calls.

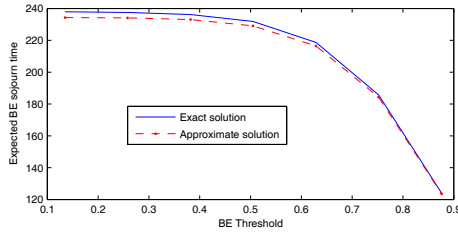


Figure 2: Expected sojourn time of BE calls versus capacity threshold of BE for  $\epsilon = 10^{-3}$ ,  $\lambda_{BE}^\epsilon = 0.209\epsilon$ ,  $\mu_{BE}^\epsilon = 10^{-5}\epsilon$ ,  $\lambda_{RT} = 1.5573$ ,  $\mu_{RT} = 1$  and any number of RT ( $M_{RT}$ ). Comparison of two methods: spectral analysis (exact solution) and singular perturbation (approximate solution).

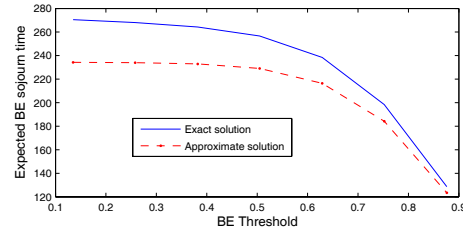


Figure 3: Expected sojourn time of BE calls versus capacity threshold of BE for  $\epsilon = 10^{-3}$ ,  $\lambda_{BE}^\epsilon = 0.209\epsilon$ ,  $\mu_{BE}^\epsilon = 10^{-5}\epsilon$ ,  $\lambda_{RT} = 0.1557$ ,  $\mu_{RT} = 0.1$  and any number of RT. Comparison of two methods: spectral analysis (exact solution) and singular perturbation (approximate solution).

## 5 Conclusion

In this paper, we considered a WCDMA system with two types of calls : real time (RT) calls that have dedicated resources, and data on real time (BE) calls (i.e. best effort) that share system capacity. We considered reservation of some capacity resources for the BE traffic as well as any capacity left over from RT calls. Our analysis approach is based on modeling of the system as a two dimensional Markov chain, where the first correspond to the number of RT calls and the second to the number of BE calls in the system. We used an approximation based on singular perturbation approach to approach the steady state of this Markov chain. We showed that our approach gives a good approximation of steady state with simple computations. However, the system can use this approach to evaluate in real time the performance measures of RT calls and BE calls and decide which best configuration or admission control improves the bandwidth utilization.

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