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Existence and Uniqueness of an Entropy Solution for Bürgers Equation

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Abstract. This paper is concerned with the proof of the existence and uniqueness of an entropy solution for Bürgers equation with additive source term (outside forces act on fluid particles). Some estimates are given.

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1. **Introduction**

Conservation laws for nonlinear hyperbolic problems rise in applied science as fluid mechanics ([8], [12]), electrochemical engineeringg, semi-conductors theory,... Systems of hyperbolic equations are systems of partial differential equations which govern a large classes of applied problems among these last appear the most fundamental problems. For example, wave equations, Euler equations of gas dynamics ([8], [12])...

Several Studies ([1], [2], [10]) have been doing on the conservation law problems where outside forces are zeros (particles motion of uniform fluid).

Consider the following problem:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = f(x,\, t) + \tau \frac{\partial^2 u}{\partial x^2}$$

It is well known that this problem admits an unique solution ([3], [5], [7]) $u_\tau \in C^3(]a,\, b[\times]0, T[)$. When $\tau \to 0$, we arrive to study the following Bürgers equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = f(x,\, t),$$

where u denotes velocity of particule and f is source function (outside force). This problem does not admit an unique solution. Beyond some critical time t_0 , the classical solution does not exist. There exists at least a weak solution defined for the time $t > t_0$. Nevertheless, the class of weak solutions is very large to ensure the uniqueness solution. We will choose an entropy unique solution among all weak solutions.

This paper is concerned with the proof of the existence and uniqueness of an entropy solution for Bürgers equation with additive source term (outside forces act on fluid particles). Some estimations are given.

2. Existence and Uniqueness of an Entropy Solution

We will prove the existence and uniqueness of an entropy solution of Bürgers equation for $x \in]a,\, b[$ and $t > 0$. Find u defined on $]a,\, b[\times]0,\, \infty[$ and verified

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial (u^2)}{\partial x} = f(x,\, t),\ (x,\, t) \in]a,\, b[\times]0,\, \infty[\\ \\ \begin{cases} u(x,0) = u_0(x), & x \in]a,\, b[\\ u(x,T) = u_T(x), & x \in]a,\, b[\\ u(a,\, t) = u_1(t), & t \in]0,\, \infty[\\ u(b,\, t) = u_2(t), & t \in]0,\, \infty[\end{cases} \end{cases} \tag{2.1}$$

Suppose

$$f \in C^2(]a,\, b[\times]0,\, \infty[) \cap L^1(]a,\, b[\times]0,\, \infty[),\ \ u_0 \in C^2(]a,\ b[),$$

$$u_1 \text{ and } \ u_2 \in C^2(]0,\, \infty[) \cap L^\infty(]0,\, \infty[),\ \ u_1(0) = u_0(a) \ \text{ and } \ u_2(0) = u_0(b).$$

To prove the existence theorem, we use a viscosity method ([1], [6]) based on the following equation:

$$\frac{\partial u_\tau}{\partial t} + \frac{1}{2}\frac{\partial ({u_\tau}^2)}{\partial x} = f(x,\, t) + \tau \frac{\partial^2 u_\tau}{\partial x^2} \tag{2.2}$$

with $\tau > 0$, τ tends to zero, and same initial and boundary data.

By an argument of compactness, we will obtain the convergence (in L^1) of an extract sequence $\{u_{\tau_n}\}$. This last tends to a solution u of (2.1) as $\tau_n \to 0,$ and characterized by a particular formulation which assure the uniqueness of solution.

Notations:

- We will use the bounded variation functions space on an open set Ω, denoted $BV(\Omega)$ ([11]) defined by

$$u \in BV(\Omega) \Leftrightarrow \|u\| = \sup_{\phi\in\Lambda}(\int\limits_{\Omega} u div\phi \; dx) < +\infty$$

where

$$\Lambda = \{\phi \in \mathcal{D}(\Omega) : |\phi|_{\infty} \leq 1\}.$$

- We take $\Omega = I$ in a space of dimension 1. We have

$$u \in BV(\Omega) \Leftrightarrow \|u\| = \lim_{(\mu\to 0)}(\frac{1}{\mu}\int_a^{b-\mu} |u(x+\mu) - u(x)| \, dx) < +\infty.$$

Denote $\Delta(I)$ the set of discrete subdivisions of I, namely the set of creasing finite sequences $\{x_0 = a, \, x_1, ..., \, x_N = b\}$. Then

$$u \in BV(\Omega) \Leftrightarrow \|u\| = \sup_{\{x_i\}\in\Delta(I)}(\sum_{i=1}^{N}|u(x_{i+1}) - u(x_i)|) < +\infty.$$

- $\|u\|$ is a semi norm on the space of $BV(\Omega)$ called total variation of u.
- $BV(\Omega)$ is a Banach space in adding $|u|_{\infty}$.
- Put $\Omega =]a, \, b[\times]0, \, T[$, with $T > 0$, we have

$$W^{1,1}(]a, \, b[\times]0, \, T[) \subset BV(]a, \, b[\times]0, \, T[).$$

For a space of dimension 1, we have

$$BV(]a, \, b[) \subset L^{\infty}(]a, \, b[).$$

- A function u of $BV(]a, \, b[)$ admits in any point a right and left limits.
- Let $u \in BV(\Omega)$; for almost any $t \in]0, \, T[$, $u(., \, t) \in BV(]a, \, b[)$ and so admits left and right limits in all point. This enables to prove existence of traces at a and b. (these points are achieved for a convergence almost every where) and thus in $L^1(]0, \, T[)$ by dominated Lebesgue theorem of convergence.
- In the same way, for almost any $x \in]a, \, b[$, $u(t, \, .) \in BV(]0, \, T[)$ and admits left and right limits in all point. Then u admits a trace at $t = 0$, achieved for a convergence almost every where and thus in $L^1(]a, \, b[)$ as $t \to 0$.

Theorem 1. *([1])(Riesz-Tamakin) The injection of $W^{1,1}(\Omega)$ in $L^1(\Omega)$ is relatively compact.*

Theorem 2. *([1], [10])(Helly) The injection of $BV(\Omega) \cap L^{\infty}(\Omega)$ in $L^1(\Omega)$ is relatively compact.*

So, We will extract a convergente sequence in $L^1(\Omega)$ from all bounded family of $W^{1,1}(\Omega)$. Moreover, a limit of this sequence is an element of $BV(\Omega) \cap L^{\infty}(\Omega)$.

Lemma 3. *([9], [1])(Saks) Let Ω be an open set of$\mathbb{R}^n$, and $v \in C^1(\Omega)$. Then*

$$\lim_{(\eta \to 0, \eta > 0)} \int_{\{x:|x|<\eta\}} |\nabla v| \, dx = 0.$$

We introduce the sign function for $\eta > 0$, $z \in \mathbb{R}$,as

$$sg_{\eta}(z) = \begin{cases} 1 & \text{if} \quad z > \eta \\ \frac{z}{\eta} & \text{if} \quad -\eta \leq \ z \leq \eta \\ -1 & \text{if} \quad z < -\eta, \end{cases}$$

moreover

$$\lim_{(\eta \to 0)} sg_{\eta}(z) = sg_0(z) = \begin{cases} 1 & \text{if} \quad z > \eta \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < -\eta \end{cases}$$

and a function ab_{η} defined on $\mathbb{R}$ by

$$ab_{\eta}(x) = \int_0^x sg_{\eta}(x) dx$$

$$\frac{\partial ab_{\eta}(x)}{\partial x} = sg_{\eta}(x)$$

$$\frac{\partial ab_{\eta}(y(x))}{\partial x} = \frac{\partial ab_{\eta}(y(x))}{\partial y} \frac{\partial y}{\partial x} = \frac{\partial y}{\partial x} sg_{\eta}(y)$$

approachs the absolute value under limit

$$\lim_{(\eta \to 0)} ab_{\eta}(x) = ab_0(x) = |x| \, .$$

Theorem 4. *The problem* (2.1) *admits an unique solution $u \in BV(]a, \, b[\times]0, \, T[) \cap L^{\infty}(]a, \, b[\times]0, \, T[)$ characterized by*

$$\int_0^T \int_a^b (|u-k| \frac{\partial \varphi}{\partial t} + sg_0(u-k)((\frac{u^2-k^2}{2}) \frac{\partial \varphi}{\partial x} + f(x,t)\varphi)) dx dt + \int_a^b |u_0(x)-k| \, \varphi(x,0) dx$$

$$\geq \int_0^T sg_0(u_2(t)-k)(\frac{u_2^2(t)-w_{ub}(t)^2}{2}) \varphi(b,t) dt - \int_0^T sg_0(u_1-k)(\frac{u_1^2(t)-w_{ua}(t)^2}{2}) \varphi(a, \, t) dt.$$

Lemma 5. *We have*

$$|u_\tau|_{L^\infty(]a,b[\times]0,T[)} \le M_0 = Max(|u_1|_{L^\infty(]0,T[)}, |u_2|_{L^\infty(]0,T[)}, |u_0|_{L^\infty(]a,b[)}, |u_T|_{L^\infty(]a,b[)}).$$

Proof. If u_τ appears as maximum at $(x_0, t_0) \in]a, b[\times]0, T[$, we will have in this point

$$\frac{\partial u_\tau}{\partial x}(x_0, t_0) = 0, \quad \frac{\partial^2 u_\tau}{\partial x^2}(x_0, t_0) \le 0$$

then (by (2.2))

$$\frac{\partial u_\tau}{\partial t}(x_0, t_0) \le f(x_0, t_0)$$

a) If

$$f(x_0, t_0) \le 0 \Rightarrow \frac{\partial u_\tau}{\partial t}(x_0, t_0) \le 0.$$

Such maximum does not appear therefore inside of $]a, b[\times]0, T[$. It appears therefore at $x = a$, at $x = b$ or at $t = 0$. Thus we have an estimate

$$|u_\tau|_{L^\infty(]a,b[\times]0,T[)} \le Max(|u_1|_{L^\infty(]0,T[)}, |u_2|_{L^\infty(]0,T[)}, |u_0|_{L^\infty(]a,b[)}) \tag{2.3}$$

b) If

$$0 \le \frac{\partial u_\tau}{\partial t}(x_0, t_0) \le f(x_0, t_0),$$

such maximum does not appear therefore inside of $]a, b[\times]0, T[$. It appears therefore at $x = a$, at $x = b$ or at $t = T$. Thus we have an estimate

$$|u_\tau|_{L^\infty(]a,\, b[\times]0,\, T[)} \le Max(|u_1|_{L^\infty(]0,T[)}, |u_2|_{L^\infty(]0,T[)}, |u_T|_{L^\infty(]a,\, b[)})$$

In any case we have the following estimate:

$$|u_\tau|_{L^\infty(]a,b[\times]0,T[)} \le M_0 = Max(|u_1|_{L^\infty(]0,T[)}, |u_2|_{L^\infty(]0,T[)}, |u_0|_{L^\infty(]a,b[)}, |u_T|_{L^\infty(]a,b[)}).$$

∎

3. **Priori Estimates**

Let $\tau > 0\,(\tau \to 0)$, $T > 0$ $(T \to \infty)$. Consider the problem (2.2) with same initial and boundary conditions for $(x, t) \in]a, b[\times]0, T[$. We know such problem admits an unique solution $u_\tau \in C^3(]a, b[\times]0, T[)$.

Theorem 6. *(estimate of $\frac{\partial u_\tau}{\partial x}$ in $L^1(]a, b[\times]0, T[)$) We have the following estimate:*

$$\left|\frac{\partial u_\tau(., t)}{\partial x}\right|_{L^1(]a,b[)} \le M_1(T).$$

Proof. For $t \in]0, T[$, (t fixe), we derive (2.2) relate to x, multiply by $sg_\eta(\frac{\partial u_\tau}{\partial x})$ and integrate on $]a, b[$. We obtain

$$(3.1)\quad \int_a^b (sg_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial}{\partial x}(\frac{\partial u_\tau}{\partial t} + \frac{1}{2}\frac{\partial ({u_\tau}^2)}{\partial x}))dx = \int_a^b (sg_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial}{\partial x}(f(x,t) + \tau\frac{\partial^2 u_\tau}{\partial x^2}))dx$$

$$\Longrightarrow \int_b^b (sg_\eta(\tfrac{\partial u_\tau}{\partial x})(\tfrac{\partial^2 u_\tau}{\partial t\partial x} + \tfrac{1}{2}\tfrac{\partial^2({u_\tau}^2)}{\partial x^2}))dx = \int_a^b (sg_\eta(\tfrac{\partial u_\tau}{\partial x})(\tfrac{\partial f(x,t)}{\partial x} + \tau\tfrac{\partial^3 u_\tau}{\partial x^3}))dx$$

$$\Longrightarrow \int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2 u_\tau}{\partial t\partial x}dx + \tfrac{1}{2}\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2({u_\tau}^2)}{\partial x^2}dx = \int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial f(x,t)}{\partial x}dx + \tau\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^3 u_\tau}{\partial x^3}dx$$

Put

$$I_1 = \int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2 u_\tau}{\partial t\partial x}dx, \qquad I_2 = \tfrac{1}{2}\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2({u_\tau}^2)}{\partial x^2}dx$$

$$I_3 = \int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial f(x,t)}{\partial x}dx, \qquad I_4 = \tau\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^3 u_\tau}{\partial x^3}dx.$$

We have

$$\begin{aligned} I_1 &= \int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2 u_\tau}{\partial t\partial x}dx = \int_a^b \tfrac{\partial}{\partial t}(\tfrac{\partial u_\tau}{\partial x})sg_\eta(\tfrac{\partial u_\tau}{\partial x})dx \\ &= \int_a^b \tfrac{\partial}{\partial t}(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial}{\partial(\frac{\partial u_\tau}{\partial x})}ab_\eta(\tfrac{\partial u_\tau}{\partial x})dx = \int_a^b \tfrac{\partial}{\partial t}(ab_\eta(\tfrac{\partial u_\tau}{\partial x}))dx = \tfrac{\partial}{\partial t}\int_a^b ab_\eta(\tfrac{\partial u_\tau}{\partial x})dx \end{aligned}$$

$$I_2 = \tfrac{1}{2}\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2({u_\tau}^2)}{\partial x^2}dx = (sg_\eta(\tfrac{\partial u_\tau}{\partial x})u_\tau\tfrac{\partial u_\tau}{\partial x})\Big|_a^b - \int_a^b \tfrac{\partial^2 u_\tau}{\partial x^2}sg'_\eta(\tfrac{\partial u_\tau}{\partial x})u_\tau\tfrac{\partial u_\tau}{\partial x}dx$$

$$|I_3| = \left|\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial f(x,t)}{\partial x}dx\right| \le \int_a^b \left|\tfrac{\partial f(x,t)}{\partial x}\right| dx$$

$$\begin{aligned} I_4 &= \tau\int_a^b sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^3 u_\tau}{\partial x^3}dx = (\tau sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^3 u_\tau}{\partial x^3})\Big|_a^b - \tau\int_a^b sg'_\eta(\tfrac{\partial u_\tau}{\partial x})(\tfrac{\partial^2 u_\tau}{\partial x^2})^2dx \\ &= (\tau sg_\eta(\tfrac{\partial u_\tau}{\partial x})\tfrac{\partial^2 u_\tau}{\partial x^2})\Big|_a^b - \tfrac{\tau}{\eta}\int_{\{x:|\frac{\partial u_\tau}{\partial x}|<\eta\}} (\tfrac{\partial^2 u_\tau}{\partial x^2})^2dx. \end{aligned}$$

Then

$$(3.1) \Rightarrow \frac{\partial}{\partial t}\int_a^b ab_\eta(\frac{\partial u_\tau}{\partial x})dx + (sg_\eta(\frac{\partial u_\tau}{\partial x})u_\tau\frac{\partial u_\tau}{\partial x})\Big|_a^b - \int_a^b \frac{\partial^2 u_\tau}{\partial x^2} sg'_\eta(\frac{\partial u_\tau}{\partial x})u_\tau\frac{\partial u_\tau}{\partial x}dx =$$
$$(sg_\eta(\frac{\partial u_\tau}{\partial x})f(x,\,t))\Big|_a^b - \frac{1}{\eta}\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \frac{\partial^2 u_\tau}{\partial x^2}f(x,\,t)dx + (\tau sg_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial^2 u_\tau}{\partial x^2})\Big|_a^b$$
$$-\frac{\tau}{\eta}\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} (\frac{\partial^2 u_\tau}{\partial x^2})^2dx$$

$$\Rightarrow \frac{\partial}{\partial t}\int_a^b ab_\eta(\frac{\partial u_\tau}{\partial x})dx + (sg_\eta(\frac{\partial u_\tau}{\partial x})(u_\tau\frac{\partial u_\tau}{\partial x} - f(x,\,t) - \tau\frac{\partial^2 u_\tau}{\partial x^2}))\Big|_a^b$$
$$= \int_a^b \frac{\partial^2 u_\tau}{\partial x^2} sg'_\eta(\frac{\partial u_\tau}{\partial x})u_\tau\frac{\partial u_\tau}{\partial x}dx - \frac{1}{\eta}\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \frac{\partial^2 u_\tau}{\partial x^2}f(x,\,t)dx - \frac{\tau}{\eta}\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} (\frac{\partial^3 u_\tau}{\partial x^3})^2dx$$
$$\Rightarrow \frac{\partial}{\partial t}\int_a^b ab_\eta(\frac{\partial u_\tau}{\partial x})dx = -\,(sg_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial u_\tau}{\partial t})\Big|_a^b + \int_a^b \frac{\partial^2 u_\tau}{\partial x^2} sg'_\eta(\frac{\partial u_\tau}{\partial x})u_\tau\frac{\partial u_\tau}{\partial x}dx$$
$$-\frac{1}{\eta}\int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \frac{\partial^2 u_\tau}{\partial x^2}f(x,\,t)dx - \frac{\tau}{\eta}\int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} (\frac{\partial^3 u_\tau}{\partial x^3})^2dx.$$

Since

$$\int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} (\frac{\partial^3 u_\tau}{\partial x^3})^2dx \geq 0,$$

then

$$(3.1) \qquad \Rightarrow \frac{\partial}{\partial t}\int_a^b ab_\eta(\frac{\partial u_\tau}{\partial x})dx \leq (-sg_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial u_\tau}{\partial t})\Big|_a^b + \int_a^b \frac{\partial^2 u_\tau}{\partial x^2} sg'_\eta(\frac{\partial u_\tau}{\partial x})u_\tau\frac{\partial u_\tau}{\partial x}dx$$
$$-\frac{1}{\eta}\int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \frac{\partial^2 u_\tau}{\partial x^2}f(x,\,t)dx$$

Moreover,

$$\left|\int_b^b sg'_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial^2 u_\tau}{\partial x^2}u_\tau\frac{\partial u_\tau}{\partial x}dx\right| \leq \int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \left|\frac{\partial^2 u_\tau}{\partial x^2}\right|\left|\frac{\frac{\partial u_\tau}{\partial x}}{\eta}\right|\left|u_\tau\right|dx$$

$$\leq M_0\int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \left|\frac{\partial^2 u_\tau}{\partial x^2}\right|\left|\frac{\frac{\partial u_\tau}{\partial x}}{\eta}\right|dx \leq M_0\int_{\left\{\,x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \left|\frac{\partial^2 u_\tau}{\partial x^2}\right|dx$$

since $\left|\frac{\frac{\partial u_\tau}{\partial x}}{\eta}\right| \leq 1$ and $\int\limits_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \left|\frac{\partial^2 u_\tau}{\partial x^2}\right| dx \to 0$ as $\eta \to 0$ (by Saks lemma),

$$(-sg_\eta(\frac{\partial u_\tau}{\partial x})\frac{\partial u_\tau}{\partial t})\Big|_a^b \leq \left|\frac{\partial u_1(t)}{\partial t}\right| + \left|\frac{\partial u_2(t)}{\partial t}\right|$$

We take a limit as $\eta \to 0$, then

$$(3.1) \quad \Rightarrow \lim_{(\eta\to 0)} \frac{\partial}{\partial t}\int\limits_a^b ab_\eta(\frac{\partial u_\tau}{\partial x})dx \leq$$

$$\lim_{(\eta\to 0)}(\left|\frac{\partial u_1(t)}{\partial t}\right| + \left|\frac{\partial u_2(t)}{\partial t}\right| + M_0 \int\limits_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \left|\frac{\partial^2 u_\tau}{\partial x^2}\right| dx - \frac{1}{\eta}\int\limits_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}} \frac{\partial^2 u_\tau}{\partial x^2} f(x,\, t)dx)$$

$$\Rightarrow \frac{\partial}{\partial t}\int\limits_a^b \left|\frac{\partial u_\tau}{\partial x}\right| dx \leq \left|\frac{\partial u_1(t)}{\partial t}\right| + \left|\frac{\partial u_2(t)}{\partial t}\right| \Rightarrow \frac{\partial}{\partial t}\left|\frac{\partial u_\tau}{\partial x}\right|_{L^1(]a,\,b[)} \leq \left|\frac{\partial u_1(t)}{\partial t}\right| + \left|\frac{\partial u_2(t)}{\partial t}\right|$$

It holds the following estimate:

$$\left|\frac{\partial u_\tau(.,t)}{\partial x}\right|_{L^1(]a,\,b[)} \leq \left|\frac{\partial u_\tau(.,0)}{\partial x}\right|_{L^1(]a,\,b[)} + \int\limits_0^t (\left|\frac{\partial u_1(t)}{\partial t}\right| + \left|\frac{\partial u_2(t)}{\partial t}\right|)dt$$

$$\leq \left|\frac{\partial u_\tau(.,0)}{\partial x}\right|_{L^1(]a,b[)} + \int\limits_0^T (\left|\frac{\partial u_1(t)}{\partial t}\right| + \left|\frac{\partial u_2(t)}{\partial t}\right|)dt \leq M_1(T).$$

Therefore

$$\left|\frac{\partial u_\tau(.,\,t)}{\partial x}\right|_{L^1(]a,\,b[)} \leq M_1(T). \tag{3.2}$$

∎

Theorem 7. *(estimate of $\frac{\partial u_\tau}{\partial t}$ in $L^1(]a,\, b[\times]0,\, T[)$) We have the following estimate:*

$$\left|\frac{\partial u_\tau}{\partial t}\right|_{L^1(]a,\,b[)} \leq M_4(T).$$

Proof. We introduce the following function:

$$\phi(x,\, t) = u_1(t)\frac{x-b}{a-b} + u_2(t)\frac{x-a}{b-a}$$

and put

$$u_\tau(x,\, t) = v_\tau(x,\, t) + \phi(x,\, t)$$

to obtain homogeneous boundary conditions. Then, a function $v_\tau(x,\, t)$ verifies equation

$$\frac{\partial(v_\tau+\phi)}{\partial t}+\frac{1}{2}\frac{\partial(v_\tau+\phi)^2}{\partial x}=f(x,\,t)+\tau\frac{\partial^2(v_\tau+\phi)}{\partial x^2},$$

and since

$$\frac{\partial^2\phi}{\partial x^2}=0$$

then we have

$$\frac{\partial v_\tau}{\partial t}+\frac{1}{2}\frac{\partial(v_\tau+\phi)^2}{\partial x}=-\frac{\partial\phi}{\partial t}+f(x,\,t)+\tau\frac{\partial^2 v_\tau}{\partial x^2} \tag{3.3}$$

with initial and boundary conditions

$$\begin{array}{l} v_\tau(a,\,t)=0\\ v_\tau(b,\,t)=0\\ v_\tau(x,0)=u_0(x)-\phi(x,0)\end{array}$$

we derive (3.3) relate to t, multiply by $sg_\eta(\frac{\partial v_\tau}{\partial t})$ and integrate on $]a,\,b[$. We obtain

$$\begin{array}{l}\frac{\partial}{\partial t}\int_a^b ab_\eta(\frac{\partial v_\tau}{\partial t})dx+\frac{1}{2}\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2(v_\tau+\phi)^2}{\partial x\partial t}dx=\\ -\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2\phi}{\partial t^2}dx+\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial f(x,t)}{\partial t}dx+\tau\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^3 v_\tau}{\partial x^2\partial t}dx\end{array} \tag{3.4}$$

Put

$$In_1=\frac{1}{2}\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2(v_\tau+\phi)^2}{\partial x\partial t}dx,\qquad In_2=\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2\phi}{\partial t^2}dx$$

$$In_3=\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial f(x,t)}{\partial t}dx,\qquad In_4=\tau\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^3 v_\tau}{\partial x^2\partial t}dx$$

We have

$$\begin{aligned} In_1&=\frac{1}{2}\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2(v_\tau+\phi)^2}{\partial x\partial t}dx\\ &=(\frac{1}{2}sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial(v_\tau+\phi)^2}{\partial t})\Big|_a^b-\frac{1}{2}\int_a^b sg'_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2 v_\tau}{\partial t\partial x}\frac{\partial(v_\tau+\phi)^2}{\partial t}dx\end{aligned}$$

and since $v_\tau(a,t)=0$ and $v_\tau(b,t)=0$ then $\left(\frac{1}{2}sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial(v_\tau+\phi)^2}{\partial t}\right)\Big|_a^b=0$. Therefore, one has

$$In_1=-\frac{1}{2}\int_a^b sg'_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2 v_\tau}{\partial t\partial x}\frac{\partial(v_\tau+\phi)^2}{\partial t}dx$$

$$=-\int_a^b sg'_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial v_\tau}{\partial t}\frac{\partial^2 v_\tau}{\partial t\partial x}(v_\tau+\phi)dx-\int_a^b sg'_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2 v_\tau}{\partial t\partial x}(v_\tau+\phi)\frac{\partial\phi}{\partial t}dx$$

$$=-\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}}\frac{\frac{\partial v_\tau}{\partial t}}{\eta}\frac{\partial v_\tau}{\partial t}\frac{\partial^2 v_\tau}{\partial t\partial x}(v_\tau+\phi)dx-\left(sg_\eta(\frac{\partial v_\tau}{\partial t})(v_\tau+\phi)\frac{\partial\phi}{\partial t}\right)\Big|_a^b$$
$$+\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial}{\partial x}((v_\tau+\phi)\frac{\partial\phi}{\partial t})dx$$

$$=-\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}}\frac{\frac{\partial v_\tau}{\partial t}}{\eta}\frac{\partial v_\tau}{\partial t}\frac{\partial^2 v_\tau}{\partial t\partial x}(v_\tau+\phi)dx+\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})(\frac{\partial}{\partial x}(v_\tau+\phi)\frac{\partial\phi}{\partial t}+(v_\tau+\phi)\frac{\partial^2\phi}{\partial t\partial x})dx$$

$$=-\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}}\frac{\frac{\partial v_\tau}{\partial t}}{\eta}\frac{\partial v_\tau}{\partial t}\frac{\partial^2 v_\tau}{\partial t\partial x}(v_\tau+\phi)dx+\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})(\frac{\partial u_\tau}{\partial x}\frac{\partial\phi}{\partial t}+u_\tau\frac{\partial^2\phi}{\partial t\partial x})dx.$$

since

$$\left|\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})(\frac{\partial}{\partial x}(u_\tau)\frac{\partial\phi}{\partial t}+(u_\tau)\frac{\partial^2\phi}{\partial t\partial x})dx\right|\le\int_a^b\left|\frac{\partial u_\tau}{\partial x}\frac{\partial\phi}{\partial t}+u_\tau\frac{\partial^2\phi}{\partial t\partial x}\right|dx$$
$$\le\int_a^b\left|\frac{\partial\phi}{\partial t}\right|\left|\frac{\partial u_\tau}{\partial x}\right|+|u_\tau|\left|\frac{\partial^2\phi}{\partial t\partial x}\right|dx$$

$\int_{\left\{x:\left|\frac{\partial u_\tau}{\partial x}\right|<\eta\right\}}\frac{v_\tau}{\eta}\frac{\partial^2 v_\tau}{\partial t\partial x}\frac{\partial(v_\tau+\phi)}{\partial t}dx\to0$ as $\eta\to0$ (by Saks Lemma),

$$In_4=\tau\int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^3 v_\tau}{\partial x^2\partial t}dx$$

$$=\left(\tau sg_\eta(\frac{\partial v_\tau}{\partial t})\frac{\partial^2 v_\tau}{\partial x\partial t}\right)\Big|_a^b-\tau\int_a^b sg'_\eta(\frac{\partial v_\tau}{\partial t})\left|\frac{\partial^2 v_\tau}{\partial x\partial t}\right|^2dx=-\tau\int_a^b sg'_\eta(\frac{\partial v_\tau}{\partial t})\left|\frac{\partial^2 v_\tau}{\partial x\partial t}\right|^2dx\le0$$

and

$$|In_3| = \left| \int_a^b sg_\eta(\frac{\partial v_\tau}{\partial t}) \frac{\partial f(x,\, t)}{\partial t} dx \right| \leq \int_a^b \left| \frac{\partial f(x,\, t)}{\partial t} \right| dx,$$

$$|In_2| = \left| \int_a^b sg_\eta(\tfrac{\partial v_\tau}{\partial t}) \tfrac{\partial^2 \phi}{\partial t^2} dx \right| \leq \int_a^b \left| \tfrac{\partial^2 \phi}{\partial t^2} \right| dx \leq \int_a^b \left| \tfrac{\partial^2 u_1(t)}{\partial t^2} \tfrac{x-b}{a-b} + \tfrac{\partial^2 u_2(t)}{\partial t^2} \tfrac{x-a}{b-a} \right| dx$$

$$\leq \int_a^b (\left| \tfrac{\partial^2 u_1(t)}{\partial t^2} \tfrac{x-b}{a-b} \right| + \left| \tfrac{\partial^2 u_2(t)}{\partial t^2} \tfrac{x-a}{b-a} \right|) dx \leq \left| \tfrac{x-b}{a-b} \right| \int_a^b \left| \tfrac{\partial^2 u_1(t)}{\partial t^2} \right| dx + \left| \tfrac{x-a}{b-a} \right| \int_a^b \left| \tfrac{\partial^2 u_2(t)}{\partial t^2} \right| dx$$

$$\leq \int_a^b \left| \tfrac{\partial^2 u_1(t)}{\partial t^2} \right| dx + \int_a^b \left| \tfrac{\partial^2 u_2(t)}{\partial t^2} \right| dx$$

We take a limit as $\eta \to 0$ in equation (3.4), then

$$\lim_{(\eta \to 0)} \tfrac{\partial}{\partial t} \int_a^b ab_\eta(\tfrac{\partial v_\tau}{\partial t}) dx = \tfrac{\partial}{\partial t} \int_a^b \left| \tfrac{\partial v_\tau}{\partial t} \right| dx$$

$$\leq \int_a^b (\left| \tfrac{\partial \phi}{\partial t} \right| \left| \tfrac{\partial u_\tau}{\partial x} \right| + |u_\tau| \left| \tfrac{\partial^2 \phi}{\partial t \partial x} \right|) dx + \int_a^b \left| \tfrac{\partial f(x,t)}{\partial t} \right| dx + \int_a^b \left| \tfrac{\partial^2 u_1(t)}{\partial t^2} \right| dx + \int_a^b \left| \tfrac{\partial^2 u_2(t)}{\partial t^2} \right| dx \leq M(t)$$

Or for any $t \in]0, T[$, we have

$$\int_a^b \left| \frac{\partial v_\tau(x,\, t)}{\partial t} \right| dx \leq \int_0^t M(t) dt + \int_a^b \left| \frac{\partial v_\tau(x,\, 0)}{\partial t} \right| dx$$

It holds that

$$\frac{\partial v_\tau(x,\, 0)}{\partial t} = -\frac{1}{2} \frac{\partial (u_0)^2}{\partial x} - \frac{\partial \phi(x,\, 0)}{\partial t} + f(x, 0) + \tau \frac{\partial^2 u_0}{\partial x^2}$$

where

$$\frac{\partial \phi(x,\, 0)}{\partial t} = \frac{\partial u_1(t)}{\partial t} \frac{x-b}{a-b} + \frac{\partial u_2(t)}{\partial t} \frac{x-a}{b-a}$$

is bounded independently of τ. Then we can bounded $\frac{\partial \phi(x,\, 0)}{\partial t}$ by a constant $cste$. Therefore we obtain

$$\int_a^b \left| \frac{\partial v_\tau(x,\, t)}{\partial t} \right| dx \leq \int_0^t M(t) dt + \int_a^b |cste|\, dx \leq \int_0^T M(t) dt + \int_a^b |cste|\, dx \leq M_2(T)$$

and thus

$$u_\tau(x,\, t) = v_\tau(x,\, t) + \phi(x,\, t)$$

$$\Rightarrow \frac{\partial u_\tau(x,t)}{\partial t} = \frac{\partial v_\tau(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial t} \Rightarrow \left|\frac{\partial u_\tau(x,t)}{\partial t}\right| \leq \left|\frac{\partial v_\tau(x,t)}{\partial t}\right| + \left|\frac{\partial \phi(x,t)}{\partial t}\right|$$

$$\Rightarrow \int_a^b \left|\frac{\partial u_\tau(x,t)}{\partial t}\right| dx \leq \int_a^b \left|\frac{\partial v_\tau(x,t)}{\partial t}\right| dx + \int_a^b \left|\frac{\partial \phi(x,t)}{\partial t}\right| dx \Rightarrow \int_a^b \left|\frac{\partial u_\tau(x,t)}{\partial t}\right| dx \leq M_2 + \int_a^b \left|\frac{\partial \phi(x,t)}{\partial t}\right| dx$$

It holds

$$\left|\frac{\partial u_\tau}{\partial t}\right|_{L^1(]a,\, b[)} \leq M_4(T) \tag{3.5}$$

∎

4. **Passage to a Limit and Characterization**

Estimates (2.3), (3.2) and (3.5) ensure that for all $\tau > 0$, solution u_τ remains in a bounded set of $W^{1,1}(]a,\, b[\times]0,\, T[)$. Then we can apply Riesz-Tamakin Theorem 2.1 and conclude that there exists a sequence u_{τ_n}convergente in $L^1(]a,\, b[\times]0,\, T[)$. We denote $u \in BV(]a,\, b[\times]0,\, T[)$ a limit of this sequence.

Moreover, one has

$$\tau\frac{\partial^2 u_\tau}{\partial x^2} = \frac{\partial u_\tau}{\partial t} + \frac{1}{2}\frac{\partial ({u_\tau}^2)}{\partial x} - f(x,\, t)$$

integrating on $]a,\, b[$, for all $t \in]0,\, T[$ then

$$\left|\tau\frac{\partial^2 u_\tau}{\partial x^2}\right|_{L^1(]a,\, b[)} \leq \left|\frac{\partial u_\tau}{\partial t}\right|_{L^1(]a,\, b[)} + |u_\tau|_{L^\infty(]a,\, b[)}\left|\frac{\partial u_\tau}{\partial x}\right|_{L^1(]a,\, b[)} + |f(x,\, t)|_{L^1(]a,\, b[)}$$

$$\Rightarrow \left|\tau\frac{\partial^2 u_\tau}{\partial x^2}\right|_{L^1(]a,\, b[)} + \left|\frac{\partial u_\tau}{\partial x}\right|_{L^1(]a,\, b[)} \leq \left|\frac{\partial u_\tau}{\partial t}\right|_{L^1(]a,\, b[)} + \left(|u_\tau|_{L^\infty(]a,\, b[)} + \tau\right)\left|\frac{\partial u_\tau}{\partial x}\right|_{L^1(]a,\, b[)}$$

$$+ |f(x,\, t)|_{L^1(]a,\, b[)} \leq M_2(t) + (\tau + M_0)M_1(t) + cste = M^*(t)$$

We have

$$M^*(t) \leq r \Rightarrow M^*(t)^2 \leq r^2 \Rightarrow \int_0^T M^*(t)^2 dt \leq \int_0^T r^2 dt = r^2 T.$$

A function $M^*(t) \in L^2(]0,\, T[)$. This estimate lets through to a weak limit on expressions $\tau\frac{\partial u_\tau(a,\, t)}{\partial x}$.

Let $\varphi(t) \in L^2(]0,T[)$, we prolonge a function $\varphi(t)$ on $]a,\, b[\times]0,\, T[$ by putting

$$\varphi_\tau(x,\, t) = \begin{cases} \varphi(t)\frac{a+\tau-x}{\tau} & \text{if } a < x < a+\tau \\ 0 & \text{if } \ x \geq a+\tau \end{cases}$$

Remark that

$$\tau\frac{\partial\varphi_\tau(x,\, t)}{\partial x} = -\varphi(t) \quad \text{ if } \ a < x < a+\tau \quad \text{ and } \quad \varphi_\tau(a,\, t) = \varphi(t)$$

Then we integrate by parts

$$\tau\int_0^T\int_a^b \tfrac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\varphi_\tau(x,\, t)dxdt = \left(\int_0^T \tau\tfrac{\partial u_\tau}{\partial x}(x,\, t)\varphi_\tau(x,\, t)dt)\right|_a^b - \tau\int_0^T\int_a^b \tfrac{\partial u_\tau}{\partial x}(x,\, t)\tfrac{\partial\varphi_\tau(x,t)}{\partial x}dt$$

$$= \int_0^T \tau\tfrac{\partial u_\tau}{\partial x}(a,\, t)\varphi(t)dt + \tau\int_0^T\int_a^b \tfrac{\partial u_\tau}{\partial x}(x,\, t)\varphi(t)dt$$

It holds

$$\left|\int_0^T \tau\tfrac{\partial u_\tau}{\partial x}(a,\, t)\varphi(t)dt\right| \leq \left|\tau\int_0^T\int_a^b \tfrac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\varphi_\tau(x,\, t)dxdt\right| + \left|\tau\int_0^T\int_a^b \tfrac{\partial u_\tau}{\partial x}(x,\, t)\varphi(t)dxdt\right|$$

$$\leq \tau\int_0^T\int_a^b \left|\tfrac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\right| |\varphi_\tau(x,\, t)|\, dxdt + \tau\int_0^T\int_a^b \left|\tfrac{\partial u_\tau}{\partial x}(x,\, t)\right| |\varphi(t)|\, dxdt$$

$$\leq \int_0^T |\varphi(t)| \int_a^b (\tau\left|\tfrac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\right| + \tau\left|\tfrac{\partial u_\tau}{\partial x}(x,\, t)\right|)dxdt$$

$$\leq \int_0^T |\varphi(t)|\, M^*(t)dxdt \leq |\varphi(t)|_{L^2(]0,\,T[)} \quad |M^*(t)|_{L^2(]0,\,T[)}$$

So, $\tau\frac{\partial u_\tau}{\partial x}(a,\, t)$ remains bounded in $L^2(]0,\, T[)$, and from this sequence we can extract a subsequence $\tau_n\frac{\partial u_\tau}{\partial x}(a,\, t)$ which converges for a weak topology of $L^2(]0,\, T[)$. Suppose a limit of this sequence is γ_a.

We proceed in the same way at $x = b$ consider the prolongement of a function $\varphi(t)$ on $]a,\, b[\times]0, T[$ taking

$$\varphi_\tau(x,\, t) = \begin{cases} 0, & \text{if } \ x \leq b-\tau \\ \varphi(t)\frac{b-\tau+x}{\tau} & \text{if } b-\tau < x < b \end{cases}$$

Remark that

$$\varphi_\tau(b,\, t) = -\varphi(t) \qquad \text{and} \qquad \tau\frac{\partial\varphi_\tau(x,\, t)}{\partial x} = \varphi(t).$$

Then we integrate by parts

$$\tau\int_0^T\int_a^b \frac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\varphi_\tau(x,\, t)dxdt = \left(\int_0^T \tau\frac{\partial u_\tau}{\partial x}(x,\, t)\varphi_\tau(x,\, t)dt\right)\Big|_a^b - \tau\int_0^T\int_a^b \frac{\partial u_\tau}{\partial x}(x,\, t)\frac{\partial\varphi_\tau(x,t)}{\partial x}dt$$

$$= \int_0^T -\tau\frac{\partial u_\tau}{\partial x}(b,\, t)\varphi(t)dt - \tau\int_0^T\int_a^b \frac{\partial u_\tau}{\partial x}(x,\, t)\varphi(t)dt.$$

It holds that

$$\left|\int_0^T \tau\frac{\partial u_\tau}{\partial x}(b,\, t)\varphi(t)dt\right| \le \left|\tau\int_0^T\int_a^b \frac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\varphi_\tau(x,\, t)dxdt\right| + \left|\tau\int_0^T\int_a^b \frac{\partial u_\tau}{\partial x}(x,\, t)\varphi(t)dxdt\right|$$

$$\le \tau\int_0^T\int_a^b \left|\frac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\right| |\varphi_\tau(x,\, t)|\, dxdt + \tau\int_0^T\int_a^b \left|\frac{\partial u_\tau}{\partial x}(x,\, t)\right| |\varphi(t)|\, dxdt$$

$$\le \int_0^T |\varphi(t)| \int_a^b \left(\tau\left|\frac{\partial^2 u_\tau}{\partial x^2}(x,\, t)\right| + \tau\left|\frac{\partial u_\tau}{\partial x}(x,\, t)\right|\right)dxdt$$

$$\le \int_0^T |\varphi(t)|\, M^*(t)dxdt \le |\varphi(t)|_{L^2(]0,\,T[)} \quad |M^*(t)|_{L^2(]0,\,T[)}\,.$$

So, $\tau\frac{\partial u_\tau}{\partial x}(b,\, t)$ remains bounded in $L^2\,(]0,\, T[)\,,$ and from this sequence we can extract a subsequence $\tau_n\frac{\partial u_\tau}{\partial x}(b,\, t)$ which converges for a weak topology of $L^2\,(]0,\, T[)\,.$ Suppose its limit is γ_b.

For any $\tau > 0$ let $\{u_{\tau_n}\}$ be the previous extract sequence. Now, we will prove u_τ is a global solution to problem (2.2).

Theorem 8. *We have*

$$\int_0^T\int_a^b ((|u-k|)\frac{\partial\varphi}{\partial t} + sg_0(u-k)((\frac{u^2-k^2}{2})\frac{\partial\varphi}{\partial x} + f(x,\, t)\varphi))dxdt + \int_a^b |u_0(x)-k|\,\varphi(x,0)dx$$

$$\ge \int_0^T sg_0(u_2(t)-k)(\frac{u_2^2(t)-\varkappa_{ub}(t)^2}{2})\varphi(b,t)dt - \int_0^T sg_0(u_1-k)(\frac{u_1^2(t)-\varkappa_{ua}(t)^2}{2})\varphi(a,\, t)dt,$$

which gives a characterization of inquired solution of (2.1) *in* $BV(]a,\, b[\times]0,\, T[)\cap L^\infty(]a,\, b[\times]0,\, T[)$ *and where* $\varkappa_{ua}(t)$ *is a trace of* $u(.,\, t)$ *at* a.

Proof. Let $k \in \mathbb{R}, \varphi \in C^2\left([a,\, b] \times [0,\, T]\right)$ such that $\varphi(x, T) = 0$ for all $x \in]a,\, b[$ and verified

$$\forall (x,\, t) \in [a,\, b] \times [0,\, T]\,, \quad \varphi(x,\, t) \geq 0.$$

Let $\eta > 0$. We multiply (2.2) by $sg_\eta(u_\tau - k)$. Introducing a function $G_{\eta,\, k}(u_\tau)$ defined by

$$\begin{cases} \frac{\partial G_{\eta,\, k}(u_\tau)}{\partial x} = sg_\eta(u_\tau - k) u_\tau \frac{\partial u_\tau}{\partial x} \\ G_{\eta,\, k}(k) = 0, \end{cases}$$

then

$$sg_\eta(u_\tau - k)\frac{\partial u_\tau}{\partial t} + sg_\eta(u_\tau - k)u_\tau \frac{\partial u_\tau}{\partial x} = sg_\eta(u_\tau - k) f(x,\, t) + \tau sg_\eta(u_\tau - k)\frac{\partial^2 u_\tau}{\partial x^2}$$

and

$$\text{(4.1)} \qquad \frac{\partial ab_\eta(u_\tau - k)}{\partial t} + \frac{\partial G_{\eta,\, k}(u_\tau)}{\partial x} = sg_\eta(u_\tau - k) f(x,\, t) + \tau sg_\eta(u_\tau - k)\frac{\partial^2 u_\tau}{\partial x^2}$$

We multiply again by $\varphi(x,\, t)$ and we integrate by parts on $]a,\, b[\, \times\,]0,\, T[$ to obtain

$$\int_0^T\!\!\int_a^b \frac{\partial ab_\eta(u_\tau - k)}{\partial t} \varphi dx dt + \int_0^T\!\!\int_a^b \frac{\partial G_{\eta,\, k}(u_\tau)}{\partial x} \varphi dx dt = \int_0^T\!\!\int_a^b sg_\eta(u_\tau - k) f(x,\, t)\varphi dx dt$$
$$+ \int_0^T\!\!\int_a^b (\tau sg_\eta(u_\tau - k)\varphi)\frac{\partial^2 u_\tau}{\partial x^2} dx dt$$

$$\Rightarrow -\int_a^b ab_\eta(u_0(x)-k)\varphi(x,0)dx - \int_0^T\int_a^b ab_\eta(u_\tau-k)\tfrac{\partial\varphi}{\partial t}dxdt$$

$$-\int_0^T\int_a^b G_{\eta,k}(u_\tau)\tfrac{\partial\varphi}{\partial x}dxdt + \int_0^T G_{\eta,k}(u_2(t))\varphi(b,\,t) - G_{\eta,k}(u_1(t))\varphi(a,\,t)dt$$

$$= \int_0^T \tau sg_\eta(u_2(t)-k)\tfrac{\partial u_\tau(b,t)}{\partial x}\varphi(b,\,t) - \tau sg_\eta(u_1(t)-k)\tfrac{\partial u_\tau(a,t)}{\partial x}\varphi(a,\,t)dt$$

$$-\int_0^T\int_a^b \tfrac{\partial}{\partial x}(\tau sg_\eta(u_\tau-k)\varphi(x,\,t))\tfrac{\partial u_\tau}{\partial x}dxdt + \int_0^T\int_a^b sg_\eta(u_\tau-k)f(x,\,t)\varphi dxdt$$

$$\Rightarrow \int_a^b ab_\eta(u_0(x)-k)\varphi(x,0)dx + \int_0^T\int_a^b ab_\eta(u_\tau-k)\tfrac{\partial\varphi}{\partial t}dxdt + \int_0^T\int_a^b sg_\eta(u_\tau-k)f(x,\,t)\varphi dxdt$$

$$+\int_0^T\int_a^b G_{\eta,k}(u_\tau)\tfrac{\partial\varphi}{\partial x}dxdt - \int_0^T G_{\eta,k}(u_2(t))\varphi(b,\,t)dt + \int_0^T G_{\eta,k}(u_1(t))\varphi(a,\,t)dt$$

$$= -\int_0^T(\tau sg_\eta(u_2(t)-k)\varphi(b,\,t))\tfrac{\partial u_\tau(b,t)}{\partial x}dt + \int_0^T(\tau sg_\eta(u_1(t)-k)\varphi(a,\,t))\tfrac{\partial u_\tau(a,t)}{\partial x}dt$$

$$+\int_0^T\int_a^b(\tau\tfrac{\partial sg_\eta(u_\tau-k)}{\partial x}\varphi(x,t))\tfrac{\partial u_\tau}{\partial x}dxdt + \int_0^T\int_a^b(\tau sg_\eta(u_\tau-k)\tfrac{\partial\varphi(x,t)}{\partial x})\tfrac{\partial u_\tau}{\partial x}dxdt$$

$$\Rightarrow \int_0^T\int_a^b ab_\eta(u_\tau-k)\tfrac{\partial\varphi}{\partial t}dxdt + \int_0^T\int_a^b sg_\eta(u_\tau-k)f(x,\,t)\varphi dxdt + \int_0^T\int_a^b G_{\eta,k}(u_\tau)\tfrac{\partial\varphi}{\partial x}dxdt$$

$$-\int_0^T\int_a^b(\tau\tfrac{\partial sg_\eta(u_\tau-k)}{\partial x}\varphi(x,\,t))\tfrac{\partial u_\tau}{\partial x}dxdt - \int_0^T\int_a^b((\tau sg_\eta(u_\tau-k)\tfrac{\partial\varphi(x,t)}{\partial x}))\tfrac{\partial u_\tau}{\partial x}dxdt$$

$$= \int_0^T(\tau sg_\eta(u_1(t)-k)\varphi(a,\,t))\tfrac{\partial u_\tau(a,t)}{\partial x}dt - \int_0^T(\tau sg_\eta(u_2(t)-k)\varphi(b,\,t))\tfrac{\partial u_\tau(b,t)}{\partial x}dt$$

$$+\int_0^T G_{\eta,k}(u_2(t))\varphi(b,\,t)dt - \int_0^T G_{\eta,k}(u_1(t))\varphi(a,\,t)dt - \int_a^b ab_\eta(u_0(x)-k)\varphi(x,0)dx.$$

$$\Rightarrow \int_0^T\int_a^b (ab_\eta(u_\tau-k)\tfrac{\partial\varphi}{\partial t}dxdt + sg_\eta(u_\tau-k)f(x,\,t)\varphi + G_{\eta,\,k}(u_\tau)\tfrac{\partial\varphi}{\partial x}$$
$$-\tau sg_\eta(u_\tau-k)\tfrac{\partial\varphi(x,\,t)}{\partial x}\tfrac{\partial u_\tau}{\partial x})dxdt - \int_0^T\int_a^b \tau\tfrac{\partial sg_\eta(u_\tau-k)}{\partial u}\varphi(x,\,t)(\tfrac{\partial u_\tau}{\partial x})^2dxdt$$
$$= \int_0^T(\tau sg_\eta(u_1(t)-k)\varphi(a,\,t))\tfrac{\partial u_\tau(a,\,t)}{\partial x}dt - \int_0^T(\tau sg_\eta(u_2(t)-k)\varphi(b,\,t))\tfrac{\partial u_\tau(b,\,t)}{\partial x}dt$$
$$+\int_0^T G_{\eta,\,k}(u_2(t))\varphi(b,\,t)dt - \int_0^T G_{\eta,\,k}(u_1(t))\varphi(a,\,t)dt - \int_a^b ab_\eta(u_0(x)-k)\varphi(x,0)dx$$

and since

$$\int_a^b\int_0^T(\tau\frac{\partial sg_\eta(u-k)}{\partial u_\tau}\varphi(x,\,t))(\frac{\partial u}{\partial x})^2dxdt \geq 0,$$

we have inequality

$$\int_0^T\int_a^b (ab_\eta(u_\tau-k)\tfrac{\partial\varphi}{\partial t}dxdt + sg_\eta(u_\tau-k)f(x,t)\varphi + G_{\eta,\,k}(u_\tau)\tfrac{\partial\varphi}{\partial x} - \tau sg_\eta(u_\tau-k)\tfrac{\partial\varphi(x,t)}{\partial x}\tfrac{\partial u_\tau}{\partial x})dxdt$$
$$\geq \int_0^T\tau sg_\eta(u_1(t)-k)\tfrac{\partial u_\tau(a,\,t)}{\partial x}\varphi(a,\,t)dt - \int_0^T\tau sg_\eta(u_2(t)-k)\tfrac{\partial u_\tau(b,\,t)}{\partial x}\varphi(b,\,t)dt$$
$$+\int_0^T G_{\eta,\,k}(u_2(t))\varphi(b,\,t)dt - \int_0^T G_{\eta,\,k}(u_1(t))\varphi(a,\,t)dt - \int_a^b ab_\eta(u_0(x)-k)\varphi(x,0)dx$$

for fixed η, we take a limit as $\tau\to 0$ (because strongly convergence of u_τ to u) and since

$$\tau\frac{\partial u_\tau}{\partial x}(a,\,t)\to\gamma_a \quad \text{and} \quad \tau\frac{\partial u_\tau}{\partial x}(b,\,t)\to\gamma_b \qquad \text{(weak convergence)},$$

then

$$\lim_{(\tau\to 0)}\int_a^b\int_0^T(\tau sg_\eta(u-k)\frac{\partial\varphi(x,\,t)}{\partial x})\frac{\partial u}{\partial x}dxdt = 0$$

because

$$\left|\int_0^T\int_a^b(\tau sg_\eta(u-k)\tfrac{\partial\varphi(x,t)}{\partial x})\tfrac{\partial u}{\partial x}dxdt\right| \leq \tau\int_a^b\int_0^T\left|\tfrac{\partial\varphi(x,t)}{\partial x}\right|\left|\tfrac{\partial u}{\partial x}\right|dxdt$$
$$\leq \tau\left|\tfrac{\partial\varphi(x,t)}{\partial x}\right|_{L^\infty(]a,b[\times]0,T[)}\left|\tfrac{\partial u}{\partial x}\right|_{L^1(]a,b[\times]0,T[)} \leq \tau\left|\tfrac{\partial\varphi(x,t)}{\partial x}\right|_{L^\infty(]a,b[\times]0,T[)}M_1(T).$$

We have

$$\int_0^T\int_a^b (ab_\eta(u_\tau - k)\tfrac{\partial\varphi}{\partial t}dxdt + sg_\eta(u_\tau - k)f(x,\, t)\varphi + G_{\eta,\,k}(u_\tau)\tfrac{\partial\varphi}{\partial x})dxdt$$

$$\geq \int_0^T sg_\eta(u_1(t)-k)\varphi(a,\, t)\gamma_a dt - sg_\eta(u_2(t)-k)\varphi(b,\, t)\gamma_b + G_{\eta,\,k}(u_2(t))\varphi(b,\, t)dt$$

$$-\int_0^T G_{\eta,\,k}(u_1(t))\varphi(a,\, t)dt - \int_a^b ab_\eta(u_0(x)-k)\varphi(x,0)dx.$$

We take a limit as $\eta \to 0$, we remark that

$$\tfrac{\partial G_{0,\,k}(u)}{\partial x} = sg_0(u-k)u\tfrac{\partial u}{\partial x} \Rightarrow \int_k^u \tfrac{\partial G_{0,\,k}(u)}{\partial x}dx = sg_0(u-k)\int_k^u u\tfrac{\partial u}{\partial x}dx$$

$$\Rightarrow G_{0,\,k}(u) - G_{0,\,k}(k) = \tfrac{sg_0(u-k)}{2}(u^2-k^2) \Rightarrow G_{0,\,k}(u) = \tfrac{sg_0(u-k)}{2}(u^2-k^2).$$

We obtain

(4.2)

$$\int_0^T\int_a^b ((|u-k|)\tfrac{\partial\varphi}{\partial t} + sg_0(u-k)((\tfrac{u^2-k^2}{2})\tfrac{\partial\varphi}{\partial x} + f(x,t)\varphi))dxdt + \int_a^b |u_0(x)-k|\,\varphi(x,0)dx$$

$$\geq \int_0^T sg_0(u_2(t)-k)(\tfrac{u_2^2-k^2}{2} - \gamma_b)\varphi(b,t)dt - \int_0^T sg_0(u_1-k)(\tfrac{u_1^2-k^2}{2} - \gamma_a)\varphi(a,\, t)dt$$

It remains to identify γ_b and γ_a. To do this, we take a particular function φ defined by

$$\varphi(x,\, t) = \varpi(t)\vartheta_\rho(x) \qquad (x,\, t) \in\,]a,\, b[\, \times\,]0, T[$$

with $\varpi(t) \in C^2(]0,T[)$ and $\varpi(t) \geq 0$ such that $\varpi(0) = 0, \varpi(T) = 0$ and $\vartheta_\rho(x) \in C^2(]a,b[)$, $\vartheta_\rho(x) \geq 0$ and $\frac{\partial\vartheta_\rho(x)}{\partial x} \leq 0$ such that

$$0 < \rho < b-a,\ \vartheta_\rho(a) = 1,\ \vartheta_\rho(a+\rho) = 0,$$

and $\lim\limits_{(\rho\to 0)} \vartheta_\rho(x) = 0$ almost every where for $x \in\,]a,\ b[$. Denote $\varkappa_{ua}(t)$ a trace of $u(.,\, t)$ at a, we take a limit as $\rho \to 0$

$$\int_0^T\int_a^b (|u-k|\,\frac{\partial\varpi(t)}{\partial t} + sg_0(u-k)f(x,\, t)\varpi(t))\vartheta_\rho(x)dxdt \to 0$$

and

$$\int_a^b |u_0(x) - k|\,\varpi(0)\vartheta_\rho(x)dx = 0$$

and

$$\int_0^T sg_0(u_2(t) - k)(\tfrac{u_2^2(t)-k^2}{2} - \gamma_b)\varpi(t)\vartheta_\rho(b)dt = 0$$

Therefore

$$\int_0^T\int_a^b sg_0(u-k)(\tfrac{u^2-k^2}{2})\tfrac{\partial\vartheta_\rho(x)}{\partial x}\varpi(t)dxdt = \int_0^T \left(sg_0(u-k)(\tfrac{u^2-k^2}{2})\vartheta_\rho(x)\varpi(t)\right)\Big|_a^b dt$$

$$-\int_0^T\int_a^b \tfrac{\partial}{\partial x}(sg_0(u-k)(\tfrac{u^2-k^2}{2})\varpi(t))\vartheta_\rho(x)dxdt.$$

Or

$$-\int_0^T\int_a^b \frac{\partial}{\partial x}(sg_0(u-k)(\frac{u^2-k^2}{2})\varpi(t))\vartheta_\rho(x)dxdt \longrightarrow 0,$$

hence

$$\int_0^T\int_a^b sg_0(u-k)(\tfrac{u^2-k^2}{2})\tfrac{\partial\vartheta_\rho(x)}{\partial x}\varpi(t)dxdt \longrightarrow$$
$$-\int_0^T sg_0(\varkappa_{ua}(t)-k)(\tfrac{\varkappa_{ua}(t)^2-k^2}{2})\vartheta_\rho(a)\varpi(t)dt.$$

Then it remains inequality

$$\int_0^T sg_0(\varkappa_{ua}(t)-k)(\tfrac{\varkappa_{ua}(t)^2-k^2}{2})\vartheta_\rho(a)\varpi(t)dt \le \int_0^T sg_0(u_1-k)(\tfrac{u_1^2(t)-k^2}{2} - \gamma_a)\varpi(t)\vartheta_\rho(a)dt$$

$$\Rightarrow \int_0^T sg_0(\varkappa_a(t)-k)(\tfrac{\varkappa_{ua}(t)^2-k^2}{2})\varpi(t)dt \le \int_0^T sg_0(u_1-k)(\tfrac{u_1^2(t)-k^2}{2} - \gamma_a)\varpi(t)dt$$

since a relation is true for all function $\varpi(t) \geq 0$, we have almost every where

$$sg_0(\varkappa_{ua}(t)-k)(\frac{\varkappa_a(t)^2-k^2}{2}) \le sg_0(u_1-k)(\frac{u_1^2(t)-k^2}{2} - \gamma_a) \tag{4.3}$$

take $k > \max(\varkappa_{ua}(t),\, u_1(t))$ and $k < \min(\varkappa_{ua}(t),\, u_1(t))$, it holds

$$\frac{\varkappa_{ua}(t)^2-k^2}{2} = \frac{u_1^2(t)-k^2}{2} - \gamma_a \Rightarrow \gamma_a = \frac{u_1^2(t)-\varkappa_{ua}(t)^2}{2}.$$

By the same way using $\vartheta_\rho(x) \geq 0$, $\frac{\partial \vartheta_\rho(x)}{\partial x} \geq 0$ such that $\vartheta_\rho(b) = 1$, $\vartheta_\rho(b-\rho) = 0$. Denote $\varkappa_{ub}(t)$ a trace of $u(., t)$ at b we obtain to a limit as $\rho \to 0$

$$\int_0^T sg_0(\varkappa_{ub}(t) - k)(\frac{\varkappa_{ub}(t)^2 - k^2}{2})\varpi(t)dt \geq \int_0^T sg_0(u_2 - k)(\frac{u_2^2(t) - k^2}{2} - \gamma_b)\varpi(t)dt.$$

Hence

$$sg_0(\varkappa_{ub}(t) - k)(\frac{\varkappa_{ub}(t)^2 - k^2}{2}) \geq sg_0(u_1 - k)(\frac{u_2^2(t) - k^2}{2} - \gamma_b). \tag{4.4}$$

Take $k > \max(\varkappa_{ub}(t), u_2(t))$ and $k < \min(\varkappa_{ub}(t), u_2(t))$, it results

$$\gamma_b = \frac{u_2^2(t) - \varkappa_{ub}(t)^2}{2}$$

Substitute γ_a and γ_b in (4.2) we obtain

(4.5)

$$\int_0^T\int_a^b (|u-k| \frac{\partial\varphi}{\partial t} + sg_0(u-k)((\frac{u^2-k^2}{2})\frac{\partial\varphi}{\partial x} + f(x,t)\varphi))dxdt + \int_a^b |u_0(x) - k|\,\varphi(x,0)dx$$

$$\geq \int_0^T sg_0(u_2(t) - k)(\frac{u_2^2(t) - \varkappa_{ub}(t)^2}{2})\varphi(b,\, t)dt - \int_0^T sg_0(u_1 - k)(\frac{u_1^2(t) - \varkappa_{ua}(t)^2}{2})\varphi(a,\, t)dt$$

A formula (4.5) gives a characterization of inquired solution of (2.1) in $BV(]a,\, b[\times]0,\, T[) \cap L^\infty(]a,\, b[\times]0,\, T[)$. Moreover, substitute γ_a in (4.3), we obtain inequality

$$\forall k \in \mathbb{R}, \quad sg_0(\varkappa_{ua}(t) - k)(\tfrac{\varkappa_{ua}(t)^2 - k^2}{2}) \leq sg_0(u_1 - k)(\tfrac{u_1^2(t) - k^2}{2} - (\tfrac{u_1^2(t) - \varkappa_{ua}(t)^2}{2}))$$

$$\Rightarrow \quad sg_0(\varkappa_{ua}(t) - k)(\tfrac{\varkappa_{ua}(t)^2 - k^2}{2}) \leq sg_0(u_1(t) - k)(\tfrac{\varkappa_{ua}(t)^2 - k^2}{2})$$

$$\Rightarrow (sg_0(u_1(t) - k) - \; sg_0(\varkappa_{ua}(t) - k))\; (\tfrac{\varkappa_{ua}(t)^2 - k^2}{2}) \geq 0$$

$$\Rightarrow sg_0(u_1(t) - \varkappa_{ua}(t))\; (\frac{\varkappa_{ua}(t)^2 - k^2}{2}) \geq 0 \tag{4.6}$$

Thus introducing interval $I(t) = [\min(u_1(t), \varkappa_{ua}(t)),\; \max(u_1(t), \varkappa_{ua}(t))]$, we have

$$\inf_{k \in I(t)} (sg_0(u_1(t) - \varkappa_{ua}(t))\; (\frac{\varkappa_{ua}(t)^2 - k^2}{2})) = 0 \tag{4.7}$$

In the same way, substitute γ_b in (4.4), we obtain inequality

$$\forall k \in \mathbb{R}, \; (sg_0(u_2(t) - k) - \; sg_0(\varkappa_{ub}(t) - k))\; (\frac{\varkappa_{ub}(t)^2 - k^2}{2}) \leq 0$$

$$sg_0(u_2(t) - \varkappa_{ub}(t)) \left(\frac{\varkappa_{ub}(t)^2 - k^2}{2}\right) \leq 0 \tag{4.8}$$

Thus introducing interval $J(t) = [\min(u_2(t), \varkappa_{ub}(t)),\ \max(u_2(t), \varkappa_{ub}(t))]$, we have

$$\sup_{k\in J(t)} (sg_0(u_1(t) - \varkappa_{ua}(t))\left(\frac{\varkappa_{ua}(t)^2 - k^2}{2}\right)) = 0. \tag{4.9}$$

Boundary conditions (4.7) and (4.9) are known Bardoux-Leroux-Nedelec conditions ([6], [10]). ∎

Theorem 9. ***(uniqueness theorem)*** *Problem* (2.1) *admits an unique solution.*

Proof. Idea proof is based on the doubling of variables technique of S. N. Kružkhov ([4], [2]). Consider two solutions u and v of (2.1) characterised by (4.5) and belonging to $BV(]a, b[\times]0, T[) \cap L^\infty(]a, b[\times]0, T[)$. We take a function $\varphi \in C_c^\infty(]a,\ b[\times]a,\ b[\times]0,\ T[\times]0,\ T[)$ in the form $\varphi(x,\ y,\ t,\ s)$ where $(x,\ y) \in (]a,\ b[)^2$ and $(t,\ s) \in (]0,\ T[)^2$ with support enclosed in $(]a,\ b[\times]0,\ T[)$. For fixed $(y,\ s)$, we choose u satisfies (4.5) with $k = v(y,\ s)$. It results that

$$\int_0^T\int_a^b (|u-v|\frac{\partial\varphi}{\partial t} + sg_0(u-v)((\frac{u^2-v^2)}{2})\frac{\partial\varphi}{\partial x} + f(x,\ t)\varphi))dxdt \geq 0$$

integrating at y and s, we obtain

$$\int_0^T\int_0^T\int_a^b\int_a^b (|u-v|\frac{\partial\varphi}{\partial t} + sg_0(u-v)((\frac{u^2-v^2}{2})\frac{\partial\varphi}{\partial x} + f(x,\ t)\varphi))dxdydtds \geq 0.$$

In the same way, v satisfies (4.5) with $k = u(x,\ t)$, it results that

$$\int_0^T\int_a^b (|v-u|\frac{\partial\varphi}{\partial s} + sg_0(v-u)((\frac{v-u^2}{2})\frac{\partial\varphi}{\partial y} + f(y,\ s)\varphi))dyds \geq 0$$

$$\Rightarrow \int_0^T\int_a^b (|v-u|\frac{\partial\varphi}{\partial s} + sg_0(u-v)((\frac{u^2-v^2}{2})\frac{\partial\varphi}{\partial y} - f(y,\ s)\varphi))dyds \geq 0$$

because $sg_0(u-v) = -sg_0(v) - u$. Integrating at x and t, we obtain

$$\int_0^T\int_0^T\int_a^b\int_a^b (|u-v|\frac{\partial\varphi}{\partial s} + sg_0(u-v)((\frac{u^2-v^2}{2})\frac{\partial\varphi}{\partial y} + f(y,\ s)\varphi))dxdydtds \geq 0.$$

Summing these two inequalities, it holds that

$$\int_0^T\int_0^T\int_a^b\int_a^b (|v-u|\,(\tfrac{\partial\varphi}{\partial t}+\tfrac{\partial\varphi}{\partial s})+sg_0(u-v)((\tfrac{u^2-v^2}{2})(\tfrac{\partial\varphi}{\partial x}+\tfrac{\partial\varphi}{\partial y})$$

$$+(f(x,t)-f(y,s))\varphi))dxdydtds\geq 0.$$

∎

We take $u=u(x,\,t)$ and $v=v(y,\,s)$, we hope estimate a quantity $|u(x,t)-v(x,t)|$.

Theorem 10. *We have the following estimate:*

$$\int_a^b |v(x,\,t)-u(x,\,t)|\,dx\leq\int_a^b |\varkappa_{v0}(x)-\varkappa_{u0}(x)|\,dx$$

where $\varkappa_{v0}(x)$ denotes a trace of $v(x,\,t)$ at $t=0$ and $\varkappa_{u0}(x)$ denotes a trace of $u(x,\,t)$ at $t=0$.

Proof. We construct a test function φ such that $(y,\,s)$ being near $(x,\,t)$ (we put unity approximation at $t-s$ and at $x-y$ in a test function). Let $\theta_\alpha\in C_c^\infty(]0,\alpha[)$ and $\lambda_\beta\in C_c^\infty(]a,\,a+\beta[)$ be two approximations of unity

$$\int_0^\alpha \theta_\alpha(r)dr=1,\qquad \int_a^{a+\beta}\lambda_\beta(r)dr=1.$$

Let $\phi\in C_c^\infty(]0,\,T[\times]a,\,b[)$ be positive. Put $\varphi(x,\,y,\,t,\,s)=\theta_\alpha(t-s)\lambda_\beta(x-y)\phi(x,\,t)$. Then, we have

$$\tfrac{\partial\varphi(x,y,t,s)}{\partial t}+\tfrac{\partial\varphi(x,y,t,s)}{\partial s}=\theta'_\alpha(t-s)\lambda_\beta(x-y)\phi(x,\,t)+\theta_\alpha(t-s)\lambda_\beta(x-y)\tfrac{\partial\phi(x,t)}{\partial t}$$

$$-\theta'_\alpha(t-s)\lambda_\beta(x-y)\phi(x,\,t)=\theta_\alpha(t-s)\lambda_\beta(x-y)\tfrac{\partial\phi(x,t)}{\partial t}$$

and by the same way

$$\frac{\partial\varphi(x,\,y,\,t,\,s)}{\partial x}+\frac{\partial\varphi(x,\,y,\,t,\,s)}{\partial y}=\theta_\alpha(t-s)\lambda_\beta(x-y)\frac{\partial\phi(x,\,t)}{\partial x}.$$

Then we obtain

$$\int_0^T\int_0^T\int_a^b\int_a^b(\theta_\alpha(t-s)\lambda_\beta(x-y))(|v-u|\,\tfrac{\partial\phi(x,t)}{\partial t}+sg_0(u-v)((\tfrac{u^2-v^2}{2})\tfrac{\partial\phi}{\partial x}\tag{4.10}$$

$$+(f(x,t)-f(y,\,s))\phi))dxdydtds\geq 0$$

We verify for each $w \in L^p(]0,\, T[\times]a,\, b[\times]0,\, T[\times]a,\, b[)$

$$\int_0^T\int_0^T\int_a^b\int_a^b w(x,\, y,\, t,\, s)(\theta_\alpha(t-s)\lambda_\beta(x-y))dxdydtds \to \int_0^T\int_a^b w(x,\, x,\, t,\, t)dxdt$$

and

$$\int_0^T\int_0^T\int_a^b\int_a^b |w(x,\, y,\, t,\, s)|\,(\theta_\alpha(t-s)\lambda_\beta(x-y))dxdydtds$$

$$= \int_0^\alpha\int_a^{a+\beta}\int_0^T\int_a^b |w(x,\, y,\, t,\, s)|\,(\theta_\alpha(t-s)\lambda_\beta(x-y))dxdydtds$$

$$= \int_0^\alpha\int_a^{a+\beta}\int_0^T\int_a^b |w(x,\, x-\xi,\, t,\, t-\zeta)|\,(\theta_\alpha(\zeta)\lambda_\beta(\xi))dxdydtds$$

$$= \int_0^\alpha\int_a^{a+\beta}\int_0^T\int_a^b |w(x,\, x-\xi,\, t,\, t-\zeta)|\,(\theta_\alpha(\zeta)\lambda_\beta(\xi))dxdydtds$$

$$\le \sup_{0<\zeta<\alpha,\ a<\xi<a+\beta}(\int_0^T\int_a^b |w(x,\, x-\xi,\, t,\, t-\zeta)|\,dxdt) = \int_0^\alpha\int_a^{a+\beta}(\theta_\alpha(\zeta)\lambda_\beta(\xi))dyds$$

this last term tends to 0 as α and β tend to 0 and hence

$$\int_0^T\int_0^T\int_a^b\int_a^b w(x,\, y,\, t,\, s)(\theta_\alpha(t-s)\lambda_\beta(x-y))dxdydtds = \int_0^T\int_a^b w(x,\, x,\, t,\, t)dxdt + w_1(\alpha,\beta)$$

where $w_1(\alpha,\beta) \to 0$ as α and β tend to 0. Expression (4.10) implies

$$\int_0^T\int_a^b(|v(x,\, t)-u(x,\, t)|\,\tfrac{\partial\phi(x,t)}{\partial t} + sg_0(u(x,\, t)$$
$$-v(x,\, t))((\tfrac{u^2(x,t)-v^2(x,t)}{2})\tfrac{\partial\phi}{\partial x}))dxdt \ge 0 \tag{4.11}$$

Now we choose a function ϕ of the form

$$\phi(x,t) = \theta(t)\Psi_\delta(x)$$

where $\theta \in C_c^2(]0,\, T[),\ \theta \ge 0$ and $\Psi_\delta \in C^2(]a,\ b[),\ \Psi_\delta \ge 0$, zero at $x=a$ and $x=b$, equal to 1 on $]a+\delta,\, b-\delta[$ with $\delta \in \left]0, \frac{b-a}{2}\right[$, fate tighten to zero. We introduce this function ϕ in (4.11), to obtain

$$\int_0^T\int_a^b |v(x,t)-u(x,t)|\,\Psi_\delta(x)\frac{\partial\theta(t)}{\partial t}dxdt + \int_0^T\int_a^b sg_0(u(x,t)-v(x,t))\left(\left(\frac{u^2(x,t)-v^2(x,t)}{2}\right)\theta(t)\frac{\partial\Psi_\delta(x)}{\partial x}\right)dxdt \geq 0$$

$$\Rightarrow \int_0^T\int_a^b |v(x,t)-u(x,t)|\,\Psi_\delta(x)\frac{\partial\theta(t)}{\partial t}dxdt = \int_0^T \left(sg_0(u(x,t)-v(x,t))\left(\frac{u^2(x,t)-v^2(x,t)}{2}\right)\chi_\delta(x)\theta(t)\right)\Big|_a^b dt = \int_0^T\int_a^b \frac{\partial(sg_0(u(x,t)-v(x,t))(\frac{u^2(x,t)-v^2(x,t)}{2})\theta(t))}{\partial x}\Psi_\delta(x)dxdt$$

$$\Rightarrow \int_0^T\int_a^b |v(x,t)-u(x,t)|\,\Psi_\delta(x)\frac{\partial\theta(t)}{\partial t}dxdt = \int_0^T sg_0(\varkappa_{ub}(t)-\varkappa_{vb}(t))\left(\frac{\varkappa_{ub}{}^2(t)-\varkappa_{vb}{}^2(t)}{2}\right)\chi_\delta(b)\theta(t)dt$$

$$-\int_0^T sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))\left(\frac{\varkappa_{ua}{}^2(t)-\varkappa_{va}{}^2(t)}{2}\right)\chi_\delta(a)\theta(t)dt$$

$$= \int_0^T\int_a^b \frac{\partial(sg_0(u(x,t)-v(x,t))(\frac{u^2(x,t)-v^2(x,t)}{2})\theta(t))}{\partial x}\chi_\delta(x)dxdt$$

where $\varkappa_{ub}(t)$ denotes a trace of u at $x=b$. (we have $\Psi_\delta(b)=\Psi_\delta(a)=0$)

$$\Rightarrow \int_0^T\int_a^b |v(x,t)-u(x,t)|\,\Psi_\delta(x)\frac{\partial\theta(t)}{\partial t}dxdt = \int_0^T\int_a^b \frac{\partial(sg_0(u(x,t)-v(x,t))(\frac{u^2(x,t)-v^2(x,t)}{2})\theta(t))}{\partial x}\Psi_\delta(x)dxdt.$$

We take a limit as $\delta \to 0$, then we obtain

$$\Rightarrow \int_0^T\int_a^b |v(x,\,t)-u(x,\,t)|\,\frac{\partial\theta(t)}{\partial t}dxdt=\int_0^T\int_a^b\frac{\partial(sg_0(u(x,t)-v(x,t))(\frac{u^2(x,t)-v^2(x,t)}{2})\theta(t))}{\partial x}dxdt$$

$$\Rightarrow \int_0^T\int_a^b |v(x,\,t)-u(x,\,t)|\,\frac{\partial\theta(t)}{\partial t}dxdt=\int_0^T\,(sg_0(u(x,\,t)-v(x,\,t))(\frac{u^2(x,t)-v^2(x,t)}{2})\theta(t))\Big|_a^b\,dt$$

$$\Rightarrow \int_0^T\int_a^b |v(x,\,t)-u(x,\,t)|\,\frac{\partial\theta(t)}{\partial t}dxdt=\int_0^T sg_0(\varkappa_{ub}(t)-\varkappa_{vb}(t))(\frac{{\varkappa_{ub}}^2(t)-{\varkappa_{vb}}^2(t)}{2})\theta(t)dt$$

$$-\int_0^T sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2})\theta(t)dt$$

retake inequality (4.6)

$$sg_0(u_1(t)-\varkappa_{ua}(t))\;(\frac{\varkappa_{ua}(t)^2-k^2}{2})\geq 0$$

with

$$k=k(t)=\begin{cases}\varkappa_{ua}(t) & \text{if } \varkappa_{ua}(t) \text{ is between } u_1(t) \text{ and } \varkappa_{va}(t)\\[2ex] u_1(t) & \text{if } u_1(t) \text{ is between } \varkappa_{ua}(t) \text{ and } \varkappa_{va}(t)\\[2ex] \varkappa_{va}(t) & \text{if } \varkappa_{va}(t) \text{ is between } u_1(t) \text{ and } \varkappa_{ua}(t).\end{cases}$$

It holds for $k=u_1(t)$,

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2})=sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-u_1^2(t)}{2})$$

$$+\,sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{u_1}^2(t)-{\varkappa_{va}}^2(t)}{2})$$

$$\Rightarrow sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2})=sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-u_1^2(t)}{2})$$

$$+sg_0(\varkappa_{va}(t)-\varkappa_{ua}(t))(\frac{{\varkappa_{va}}^2(t)-{u_1}^2(t)}{2})$$

and since $u_1(t)$ between $\varkappa_{ua}(t)$ and $\varkappa_{va}(t)$

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))=sg_0(\varkappa_{ua}(t)-u_1(t))$$

$$sg_0(\varkappa_{va}(t)-\varkappa_{ua}(t))=sg_0(\varkappa_{va}(t)-u_1(t))$$

then

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\tfrac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2}) =$$

$$sg_0(\varkappa_{ua}(t)-u_1(t))(\tfrac{{\varkappa_{ua}}^2(t)-u_1^2(t)}{2})+sg_0(\varkappa_{va}(t)-u_1(t))(\tfrac{{\varkappa_{va}}^2(t)-{u_1}^2(t)}{2})$$

according (4.6), we have

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\tfrac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2}) =$$

$$sg_0(\varkappa_{ua}(t)-u_1(t))(\tfrac{{\varkappa_{ua}}^2(t)-u_1^2(t)}{2})+sg_0(\varkappa_{va}(t)-u_1(t))(\tfrac{{\varkappa_{va}}^2(t)-{u_1}^2(t)}{2})\le 0$$

For $k=\varkappa_{ua}(t)$, we have

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\tfrac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2}) = sg_0(\varkappa_{va}(t)-\varkappa_{ua}(t))(\tfrac{{\varkappa_{va}}^2(t)-{\varkappa_{ua}}^2(t)}{2})$$

$$= sg_0(\varkappa_{va}(t)-u_1(t))(\tfrac{{\varkappa_{va}}^2(t)-{\varkappa_{ua}}^2(t)}{2})\le 0$$

For $k=\varkappa_{va}(t)$, we have

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2}) = sg_0(\varkappa_{ua}(t)-u_1(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2})\le 0.$$

It results

$$sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2})\le 0.$$

In the same way, from (4.8) and with similar choice of k, we obtain

$$sg_0(\varkappa_{ub}(t)-\varkappa_{vb}(t))(\frac{{\varkappa_{ub}}^2(t)-{\varkappa_{vb}}^2(t)}{2})\ge 0.$$

Hence

$$sg_0(\varkappa_{ub}(t)-\varkappa_{vb}(t))(\frac{{\varkappa_{ub}}^2(t)-{\varkappa_{vb}}^2(t)}{2}) - sg_0(\varkappa_{ua}(t)-\varkappa_{va}(t))(\frac{{\varkappa_{ua}}^2(t)-{\varkappa_{va}}^2(t)}{2})\ge 0$$

and then

$$\int_0^T\int_a^b |v(x,\,t)-u(x,\,t)|\,\frac{\partial\theta(t)}{\partial t}dxdt\ge 0.$$

∎

Let $t_0 \in]0, T[,\ t_1 \in]t_0, T[,\ \delta \in \left]0, \frac{t_1-t_0}{2}\right[$. Now we take a particular function θ such that

$$\theta(t) = \theta_\delta(t) = \begin{cases} > 0 & \text{if } t \in]0, T[\\ 0 & \text{outside of }]t_0, t_1[\\ 1 & \text{if } t \in]t_0+\delta, t_1-\delta[\\ \text{monotone} & \text{if } t \in]t_0, t_0+\delta[\text{ and } t \in]t_1-\delta, t_1[\end{cases}$$

retake

$$\int_0^T\int_a^b |v(x,t)-u(x,t)| \frac{\partial\theta_\delta(t)}{\partial t} dxdt = \int_{t_0}^{t_1}\int_a^b |v(x,t)-u(x,t)| \frac{\partial\theta_\delta(t)}{\partial t} dxdt$$

$$= \int_a^b \left(|v(x,t)-u(x,t)|\,\theta_\delta(t)\right)\Big|_{t_0}^{t_1} dx - \int_0^T\int_a^b \frac{\partial|v(x,t)-u(x,t)|}{\partial t}\theta_\delta(t)dxdt.$$

Since $\theta_\delta(t_0) = \theta_\delta(t_1) = 0$, then

$$\int_{t_0}^{t_1}\int_a^b |v(x,t)-u(x,t)| \frac{\partial\theta_\delta(t)}{\partial t} dxdt = -\int_{t_0}^{t_1}\int_a^b \frac{\partial\,|v(x,t)-u(x,t)|}{\partial t}\theta_\delta(t)dxdt.$$

Taking a limit as $\delta \to 0$ yields

$$\int_{t_0}^{t_1}\int_a^b |v(x,t)-u(x,t)| \frac{\partial\theta_\delta(t)}{\partial t} dxdt = -\int_{t_0}^{t_1}\int_a^b \frac{\partial|v(x,t)-u(x,t)|}{\partial t} dxdt$$

$$= -\int_a^b \left(|v(x,t)-u(x,t)|\right)\Big|_{t_0}^{t_1} dx$$

$$= \int_a^b |v(x,t_0)-u(x,t_0)|\,dx - \int_a^b |v(x,t_1)-u(x,t_1)|\,dx.$$

Therefore

$$\int_{t_0}^{t_1}\int_a^b |v(x,t)-u(x,t)| \frac{\partial\theta_\delta(t)}{\partial t} dxdt \geq 0$$

$$\Rightarrow \int_a^b |v(x,t_0)-u(x,t_0)|\,dx \geq \int_a^b |v(x,t_1)-u(x,t_1)|\,dx$$

It remains to take a limit as t_0 tends to zero and to note $t = t_1$, to obtain, for almost any $t \in]0, T[$,

$$\int_a^b |v(x,\,t) - u(x,\,t)|\,dx \le \int_a^b |\varkappa_{v0}(x) - \varkappa_{u0}(x)|\,dx$$

where $\varkappa_{v0}(x)$ denotes a trace of $v(x,\,t)$ at $t = 0$ and $\varkappa_{u0}(x)$ denotes a trace of $u(x,\,t)$ at $t = 0$. It remains to verify that $\varkappa_{u0}(x) = u_0(x) = \varkappa_{v0}(x)$ because u and v satisfy a same initial condition. We take inequality (4.5), a function φ of a form $\varphi(x,\,t) = \theta_\delta(t)\rho(x)$ where $\delta > 0, \theta_\delta(t) \in C^2(]0,T[), \theta_\delta(0) = 1, \theta_\delta(t) \ge 0$, decreasing and zero at $t = \delta$, $\rho(x) \in C^2(]a,b[), \rho(x) \ge 0$,

$$\int_0^T\int_a^b (|u-k|\,\tfrac{\partial\theta_\delta(t)}{\partial t}\rho(x) + sg_0(u-k)((\tfrac{u^2-k^2}{2})\theta_\delta(t)\tfrac{\partial\rho(x)}{\partial x} + f(x,t)\theta_\delta(t)\rho(x)))dxdt$$

$$+\int_a^b |u_0(x)-k|\,\rho(x)dx$$

$$\ge \int_0^T sg_0(u_2(t)-k)(\tfrac{u_2^2(t)-\varkappa_{ub}(t)^2}{2})\theta_\delta(t)\rho(b)dt - \int_0^T sg_0(u_1-k)(\tfrac{u_1^2(t)-\varkappa_{ua}(t)^2}{2})\theta_\delta(t)\rho(a)dt$$

$$\Rightarrow \int_0^T\int_a^b (|u-k|\,\tfrac{\partial\theta_\delta(t)}{\partial t}\rho(x) + sg_0(u-k)((\tfrac{u^2-k^2}{2})\theta_\delta(t)\tfrac{\partial\rho(x)}{\partial x} + f(x,t)\theta_\delta(t)\rho(x)))dxdt$$

$$\ge -\int_a^b |u_0(x)-k|\,\rho(x)dx.$$

Or

$$\int_a^b\int_0^T |u-k|\,\rho(x)\tfrac{\partial\theta_\delta(t)}{\partial t}dtdx = \int_a^b\int_0^\delta |u-k|\,\rho(x)\tfrac{\partial\theta_\delta(t)}{\partial t}dtdx$$

$$= \int_a^b\,(|u-k|\,\rho(x)\theta_\delta(t))|_0^\delta\,dx - \int_a^b\int_0^\delta \tfrac{\partial|u-k|\rho(x)}{\partial t}\theta_\delta(t)dtdx.$$

Then

$$\int_a^b\int_0^T |u-k|\,\rho(x)\frac{\partial\theta_\delta(t)}{\partial t}dtdx \underset{\delta\to 0}{\to} -\int_a^b |\varkappa_{u0}(x)-k|\,\rho(x)dx$$

and

$$\int_0^T\int_a^b sg_0(u-k)((\tfrac{u^2-k^2}{2})\theta_\delta(t)\tfrac{\partial\rho(x)}{\partial x}+f(x,\,t)\theta_\delta(t)\rho(x))dxdt$$

$$=\int_0^\delta\int_a^b sg_0(u-k)((\tfrac{u^2-k^2}{2})\theta_\delta(t)\tfrac{\partial\rho(x)}{\partial x}+f(x,\,t)\theta_\delta(t)\rho(x))dxdt\underset{\delta\to 0}{\to}0.$$

Hence

$$\int_0^T sg_0(u_2(t)-k)(\tfrac{u_2^2(t)-\varkappa_{ub}(t)^2}{2})\theta_\delta(t)\rho(b)dt=\int_0^\delta sg_0(u_2(t)-k)(\tfrac{u_2^2(t)-\varkappa_{ub}(t)^2}{2})\theta_\delta(t)\rho(b)dt\underset{\delta\to 0}{\to}0$$

$$\int_0^T sg_0(u_1-k)(\tfrac{u_1^2(t)-\varkappa_{ua}(t)^2}{2})\theta_\delta(t)\rho(a)dt=\int_0^\delta sg_0(u_1-k)(\tfrac{u_1^2(t)-\varkappa_{ua}(t)^2}{2})\theta_\delta(t)\rho(a)dt\underset{\delta\to 0}{\to}0$$

Taking limits as $\delta\to 0$, it holds that

$$\forall k\in\mathbb{R},\forall\rho(x)\in C^2(]a,\,b[):\int_a^b|\varkappa_{u0}(x)-k|\,\rho(x)dx\le\int_a^b|u_0(x)-k|\,\rho(x)dx$$

we deduce

$$\forall k\in\mathbb{R}:|\varkappa_{u0}(x)-k|\le|u_0(x)-k|\qquad\text{for almost any }x\in\,]a,\,b[$$

Taking $k>\max(|u_0|_{L^\infty(]a,\,b[)},|\varkappa_{u0}|_{L^\infty(]a,\,b[)})$ and $k<\min(|u_0|_{L^\infty(]a,\,b[)},|\varkappa_{u0}|_{L^\infty(]a,\,b[)})$, we obtain $\varkappa_{u0}(x)=u_0(x)$ for almost any $x\in\,]a,\,b[$, this concludes the proof of the theorem and the uniqueness of the entropy solution to problem (2.1) is proven.

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