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An Exact Solution for the Differential Equation Governing the Lateral Motion of Thin Plates Subjected to Lateral and In-Plane Loadings

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Abstract

One of the powerful analytical methods to solve partial differential equations is the Adomian decomposition method (ADM). This paper presents a novel approach for the dynamic analysis of a flexible plate by using the ADM, which is a Boundary Value Problem (BVP). In this regard, a general approach based on the generalized Fourier series expansion is applied. The obtained analytical solution is simplified in terms of a given orthogonal basis functions that these functions satisfy the boundary conditions of plate. For the first time, we solved this equation using ADM and compared the results with those of the modal classical analysis in two cases to demonstrate the validity of the present study.

Keywords: The Adomian Decomposition Method (ADM); Thin plates; Boundary Value Problems (BVPs); Orthogonal basis functions

1. Introduction

Most scientific problems and phenomena in different fields of science and engineering occur nonlinearly. Except in a limited number of these problems, we

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encounter difficulties in finding the exact analytical solutions. Decomposition methods provide the most versatile tools available in nonlinear analysis of engineering problems.

In civil and mechanical engineering sciences, for design of slabs and systems with plate behavior, it is important to know the change of deflection and stress in slabs under the different loadings. We use the classical small–deflection theory of thin plates in supplied problems.

This paper is devoted to the study of rectangular elastic plate and their governing differential equations with ADM [1-3]. In fact, we used the general solution of this differential equation from the orthogonal functions to satisfy the complex boundary conditions of plate, which depends on the nature of the supported edges.

There are restrictions for the exact analytical solution of plate and there is no general solution for any boundary conditions, and mostly, numerical solutions are applied in complex boundary condition and shape of plate.

Finally, we successfully found the general solution of the differential equation governing rectangular plates with ADM, which is the same as the classical solution.

2. Mathematical modeling of the problem

The basic differential equation of lateral motion for plates with forced, non–damped motion and subjected to lateral and in-plane loadings in classical Small–Deflection theory of thin plates is obtained [1,2]:

$$D[\nabla^4 w] = p_z(x,y,t) + n_x \frac{\partial^2 w}{\partial x^2} + n_y \frac{\partial^2 w}{\partial y^2} + 2n_{xy} \frac{\partial^2 w}{\partial x \, \partial y} - \rho h \frac{\partial^2 w}{\partial t^2} \tag{1}$$

which is a variable coefficient fourth–order parabolic partial differential equation, where $w = w(x, y, t)$ the deflection middle surface of plate or the lateral plate displacement, D is plate bending stiffness , $p_z(x, y, t)$ is lateral loads per unit area, ρ is the plate material density, n_x , n_y and n_{xy} in order in series are in-plane forces in parallel to x , y directions and shear force, and h is the plate thickness.

In general, The Homogeneous Boundary Conditions (HBCs) for rectangular plate are a combination as the followings (for example for x direction and at $x = a$) [1, 2, 3]:

$$Two\ HBCs\ for\ free\ edge(at\ x=a) \begin{cases} -D[\dfrac{\partial^3 w}{\partial x^3} + (2-v)\dfrac{\partial^3 w}{\partial y^2 \partial x}] = 0 \\ -D[\dfrac{\partial^2 w}{\partial x^2} + v\dfrac{\partial^2 w}{\partial y^2}] = 0 \end{cases} \tag{2}$$

and

$$\text{Two HBCs for fixed edge}(at\ x=a)\begin{cases} w=0 \\ \dfrac{\partial w}{\partial x}=0 \end{cases} \tag{3}$$

or

$$\text{Two HBCs for simple edge}(at\ x=a)\begin{cases} w=0 \\ -D\,[\dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2}]=0 \end{cases} \tag{4}$$

where ν the Poisson ratio of plate material.

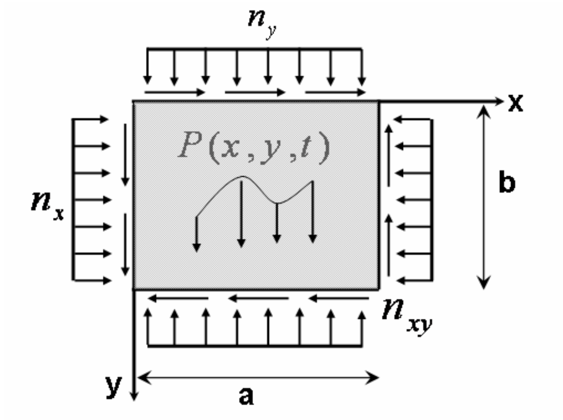

Fig. 1. The rectangular plate with general loadings

It should be noted that two HBCs must be satisfied at $x=0$, $x=a$ and two HBCs must also be satisfied at $y=0$, $y=b$. In any case for analysis of plate, we must chose two HBCs for each edge of rectangular plate that the choice as to which of the two equations has to be satisfied depends on the nature of the plate supports.

Also, it is shown that a BVP consisting of an inhomogeneous differential equation with inhomogeneous boundary conditions can be transformed into a problem consisting of an inhomogeneous differential equation with homogeneous boundary conditions [4, 5].

The transverse vibrations of plates are studies by any of the following: finite element methods, finite difference methods, and the modal analysis technique. While finite element and finite difference methods are in the category of numerical techniques, modal analysis is one of the most powerful analytical means, with the capability to join with the aforementioned methods in order to analyze the dynamic characteristics of mechanical systems.

Since the 1980s, studies have illustrated that the ADM can be applied to determine the solution of a wide range of linear and nonlinear, ordinary or partial differential and integral equations. This method gives the solution as an infinite series usually converging to an accurate solution [6-16]. In recent years, it has been applied to the problem of vibration of structural and mechanical systems in two and three space variables [17-20] as well.

In this paper, the solution of the governing equation of a uniform flexible plate is presented which takes the boundary conditions of the problem into account. For this purpose, the initial conditions are expanded using the extended Fourier series. The final solution is compared with the result from modal analysis. A comparison shows that both techniques converge to the same solution as the series approaches infinity.

3. Formulation with ADM

Using the ADM for Eq. (1) can be rewritten in operator form as:

$$\rho h[L_t w\,] + D[L_{\nabla^4} w\,] = P_z\,(x\,,y\,,t\,) + n_x\,[L_x w\,] + n_y\,[L_y w\,] + 2n_{xy}\,[L_{xy} w\,] \tag{5}$$

where the $L_t, L_{\nabla^4}, L_x, L_y, L_{xy}$ operators and $L^{-1}{}_t$ inverse are defined as follows:

$$\begin{cases} L_t w\; = \dfrac{\partial^2 w}{\partial t^2} & , L_{\nabla^4} w\; = \dfrac{\partial^4 w}{\partial x^4} + 2\dfrac{\partial^4 w}{\partial x^2 \partial y^2} + \dfrac{\partial^4 w}{\partial y^4} \\ L_x w\; = \dfrac{\partial^2 w}{\partial x^2} & , L_y w\; = \dfrac{\partial^2 w}{\partial y^2} \qquad , L_{xy} w\; = \dfrac{\partial^2 w}{\partial x\,\partial y} \end{cases} \tag{6}$$

and

$$L^{-1}{}_t w = \int_0^t \int_0^\sigma w(x,y,\tau)d\tau\; d\sigma \tag{7}$$

The general solution of an inhomogeneous linear equation (1), $w(x,y,t)$, is a sum of a general solution of the corresponding homogeneous equation $u(x,y,t)$ and a particular of the inhomogeneous equation $v(x,y,t)$ as follows:

$$w(x,y,t) = u(x,y,t) + v(x,y,t) \tag{8}$$

These two terms will be evaluated separately in the following sections.

3.1. Homogeneous problem

In order to solve the homogeneous part, using ADM, the notation of Ref. [4] is used in this section. Neglecting the source term on the right–hand side of Eq. (5) and introducing the, $L^{-1}{}_t$ operator on both sides, the solution of the homogeneous equation can be written as

$$L_t\,(u\,) = -\frac{D}{\rho h}[L_{\nabla^4} u\,] + \frac{n_x}{\rho h}[L_x u\,] + \frac{n_y}{\rho h}[L_y u\,] + 2\frac{n_{xy}}{\rho h}[L_{xy} u\,] \tag{9}$$

$$u(x,y,t) = f(x,y) + t\ g(x,y) + \\ (-1)L^{-1}{}_t\{\frac{D}{\rho h}[L_{\nabla^4}u] - \frac{n_x}{\rho h}[L_x u] - \frac{n_y}{\rho h}[L_y u] - 2\frac{n_{xy}}{\rho h}[L_{xy}u]\} \tag{10}$$

By using the Adomian decomposition method, $u(x,y,t)$ can be expanded as an infinite series expansion in terms of the $u_i(x,y,t)$ components:

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t) \tag{11}$$

In order to find the components, $u_i(x,y,t)$, substitution of Eq. (11) into both sides of Eq. (10) yields:

$$\sum_{i=0}^{\infty} u_i(x,y,t) = f(x,y) + t\ g(x,y) + \\ \left(-\frac{1}{\rho h}\right) L^{-1}{}_t \left\{ (D\ L_{\nabla^4} - n_x\ L_x - n_y\ L_y - 2n_{xy}\ L_{xy}) \left[\sum_{i=0}^{\infty} u_i(x,y,t) \right] \right\} \tag{12}$$

Considering the decomposition method, $u_0(x,y,t)$ is assumed to be of following form:

$$u_0(x,y,t) = f(x,y) + tg(x,y) \tag{13}$$

Along with the following recurrence relation for $u_i(x,y,t)$:

$$u_i(x,y,t) = \left(-\frac{1}{\rho h}\right) L^{-1}{}_t \left\{ (D\ L_{\nabla^4} - n_x\ L_x - n_y\ L_y - 2n_{xy}\ L_{xy}) [u_{i-1}(x,y,t)] \right\} \qquad i \geq 1 \tag{14}$$

Thus the first i terms of the series are

$$u_0(x,y,t) = f(x,y) + tg(x,y) \tag{15}$$

$$u_1(x,y,t) = \left(-\frac{1}{\rho h}\right) L^{-1}{}_t \left\{ (D\ L_{\nabla^4} - n_x\ L_x - n_y\ L_y - 2n_{xy}\ L_{xy}) [u_0(x,y,t)] \right\} \\ = \left(-\frac{1}{\rho h}\right) (D\ L_{\nabla^4} - n_x\ L_x - n_y\ L_y - 2n_{xy}\ L_{xy}) [\frac{t^2}{2!} f(x,y) + \frac{t^3}{3!} g(x,y)] \tag{16}$$

$$u_2(x,y,t) = \left(-\frac{1}{\rho h}\right) L^{-1}{}_t \left\{ (D\ L_{\nabla^4} - n_x\ L_x - n_y\ L_y - 2n_{xy}\ L_{xy}) [u_1(x,y,t)] \right\} \\ = \left(-\frac{1}{\rho h}\right)^2 \begin{Bmatrix} (D)^2 L^2{}_{\nabla^4} + (-n_x)^2 L^2{}_x + \\ (-n_y)^2 L^2{}_y + (-2n_{xy})^2 L^2{}_{xy} \end{Bmatrix} [\frac{t^4}{4!} f(x,y) + \frac{t^5}{5!} g(x,y)] \tag{17}$$

$\vdots$

$$u_i(x,y,t)=\left(-\frac{1}{\rho h}\right)L^{-1}_t\left\{(D\,L_{\nabla^4}-n_x\,L_x-n_y\,L_y-2n_{xy}\,L_{xy})\left[u_{i-1}(x,y,t)\right]\right\}$$

$$=\left(-\frac{1}{\rho h}\right)^i\begin{Bmatrix}(D)^i L^i{}_{\nabla^4}+(-n_x)^i L^i{}_x\\ +(-n_y)^i L^i{}_y+(-2n_{xy})^i L^i{}_{xy}\end{Bmatrix}[\frac{t^{2i}}{(2i)!}f(x,y)+\frac{t^{2i+1}}{(2i+1)!}g(x,y)]$$

$$u_i(x,y,t)=\left\{(-1)^i L^i{}_M\right\}[\frac{t^{2i}}{(2i)!}f(x,y)+\frac{t^{2i+1}}{(2i+1)!}g(x,y)] \tag{18}$$

where

$$L_M=\left\{\frac{D}{\rho h}L_{\nabla^4}-\frac{n_x}{\rho h}L_x-\frac{n_y}{\rho h}L_y-2\frac{n_{xy}}{\rho h}L_{xy}\right\} \tag{19}$$

$$L^i{}_M=\left\{(\frac{D}{\rho h})^i L^i{}_{\nabla^4}+(-\frac{n_x}{\rho h})^i L^i{}_x+(-\frac{n_y}{\rho h})^i L^i{}_y+(-2\frac{n_{xy}}{\rho h})^i L^i{}_{xy}\right\} \tag{20}$$

$$L^i{}_P=L_P L^{i-1}{}_P \tag{21}$$

Operator L_P can be every operator according up relations.

In fact, $f(x,y),g(x,y)$ in order in series are initial conditions or initial displacement and velocity of plate and then we can be written as:

$$u(x,y,0)=f(x,y) \tag{22}$$

$$\frac{\partial u}{\partial t}(x,y,0)=g(x,y) \tag{23}$$

It should be noted that the above functions $f(x,y),g(x,y)$ satisfy the boundary conditions of the problem. On the other hand, the general solution of the homogeneous equation is also a sum of the $u_i(x,y,t)$ terms. In addition, if all $u_i(x,y,t)$ functions satisfy the boundary conditions, then one may state that the sum of them also satisfies the boundary conditions. As shown in Eq. (18), $u_i(x,y,t)$ functions are determined by applying the $(-1)^i L^i{}_M$ operator to the functions $f(x,y),g(x,y)$. This may lead to $u_i(x,y,t)$ function which either are zero or do not satisfy the boundary conditions at all. To prevent this difficulty, the functions $f(x,y),g(x,y)$ are expanded in terms of the known orthogonal function $\phi_1(x,y),\phi_2(x,y),...$ as a generalization of the Fourier series expansion. $\phi_1(x,y),\phi_2(x,y),...$ can be selected to satisfy the boundary conditions before and after applying $(-1)^i L^i{}_M$ operator (see Eq. (32)). As a result, the functions $f(x,y),g(x,y)$ become

$$f(x,y)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{kj}\phi_{kj}(x,y) \tag{24}$$

$$g(x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} b_{kj}\phi_{kj}(x,y) \tag{25}$$

Where the coefficients a_{kj}, b_{kj} are given by the following relations:

$$a_{kj} = \int_{x=0}^{a}\int_{y=0}^{b} f(x,y)\phi_{kj}(x,y)\,dy\,dx \tag{26}$$

$$b_{kj} = \int_{x=0}^{a}\int_{y=0}^{b} g(x,y)\phi_{kj}(x,y)\,dy\,dx \tag{27}$$

The best set of functions for the generalized Fourier series expansion in the case of our physical problem is the set of eigenfunctions of the following self–adjoint system:

$$L_M\,\phi_{kj}(x,y) = \lambda_{kj}\phi_{kj}(x,y) \tag{28}$$

Previous studies indicate that the eigenvalue problem defined in Eq. (28) yields an infinite set of real eigenvalues and eigenfunctions $(\lambda_{kj}, \phi_{kj}(x,y))$.These eigenfunctions constitute the basis for the infinite–dimensional Hilbert space. Therefore, every function $h(x,y)$ with continuous $L_M\,h(x,y)$ which satisfies the boundary conditions of the system that can be expanded in an absolutely and uniformly convergent series in the eigenfunctions. Due to homogeneity of the eigenvalue problem, only the shape of the eigenfunctions is unique and the amplitude is arbitrary. According to Eq. (28), we can normalize the eigenfunctions using mass and stiffness operators as follows:

$$\int_{x=0}^{a}\int_{y=0}^{b} \phi_{lh}(x,y)\phi_{kj}(x,y)\,dy\,dx = \begin{cases} 0 & lh \neq kj \\ 1 & lh = kj \end{cases} \tag{29}$$

$$\int_{x=0}^{a}\int_{y=0}^{b} \phi_{lh}(x,y)L_M.\phi_{kj}(x,y)\,dy\,dx = \begin{cases} 0 & lh \neq kj \\ \lambda_{kj} & lh = kj \end{cases} \tag{30}$$

Eqs.(37) lead also to

$$L_M\phi_{kj} = \lambda_{kj}\phi_{kj} \tag{31}$$

And finally we obtain

$$L^i{}_M\,\phi_{kj} = (\lambda_{kj})^i\,\phi_{kj} \tag{32}$$

Using Eqs. (22) and (23) and substituting Eqs.(24) and (25) into Eq. (18) and embedding the result into Eq. (12) yields:

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y,t) = f(x,y) + t\,g(x,y) +$$

$$\sum_{i=1}^{\infty}\left(-\frac{1}{\rho h}\right)^i \begin{Bmatrix} (D)^i L^i{}_{\nabla^4} + (-n_x)^i L^i{}_x + \\ (-n_y)^i L^i{}_y + (-2n_{xy})^i L^i{}_{xy} \end{Bmatrix} \left[\frac{t^{2i}}{(2i)!} f(x,y) + \frac{t^{2i+1}}{(2i+1)!} g(x,y)\right]$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{kj}\phi_{kj}(x,y) + t\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} b_{kj}\phi_{kj}(x,y) +$$

$$\sum_{i=1}^{\infty}\left\{(-1)^i L^i{}_M\right\}\left[\frac{t^{2i}}{(2i)!}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{kj}\phi_{kj}(x,y) + \frac{t^{2i+1}}{(2i+1)!}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} b_{kj}\phi_{kj}(x,y)\right]$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[a_{kj}\sum_{i=0}^{\infty}\left\{(-1)^i\,\frac{(t)^{2i}(\lambda_{kj})^i}{(2i)!}\right\} + b_{kj}\sum_{i=0}^{\infty}\left\{(-1)^i\,\frac{(t)^{2i+1}(\lambda_{kj})^i}{(2i+1)!}\right\}\right]\phi_{kj}(x,y)$$

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[a_{kj}\sum_{i=0}^{\infty}\left\{(-1)^i\,\frac{(t\sqrt{\lambda_{kj}})^{2i}}{(2i)!}\right\} + \frac{b_{kj}}{\sqrt{\lambda_{kj}}}\sum_{i=0}^{\infty}\left\{(-1)^i\,\frac{(t\sqrt{\lambda_{kj}})^{2i+1}}{(2i+1)!}\right\}\right]\phi_{kj}(x,y)$$

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[a_{kj}\cos(t\sqrt{\lambda_{kj}}) + \frac{b_{kj}}{\sqrt{\lambda_{kj}}}\sin(t\sqrt{\lambda_{kj}})\right]\phi_{kj}(x,y) \tag{33}$$

3.2. Inhomogeneous problem

Neglecting the first two terms on the right–hand side of Eq. (10) and introducing the $L^{-1}{}_t$ operator on both sides of Eq. (6), the particular solution of the inhomogeneous equation can be written as

$$v(x,y,t) = (-1)L^{-1}{}_t\left[\frac{D}{\rho h}L_{\nabla^4} - \frac{n_x}{\rho h}L_x - \frac{n_y}{\rho h}L_y - 2\frac{n_{xy}}{\rho h}L_{xy}\right]v(x,y,t)$$
$$+\frac{1}{\rho h}L^{-1}{}_t P_z(x,y,t) \tag{34}$$

A similar procedure adopted in the previous section can be used to decompose the solution by an infinite sum of component expressed in a series form by

$$v(x,y,t) = \sum_{i=0}^{\infty} v_i(x,y,t) \tag{35}$$

And $v_i(x,y,t)$ can be determined in a similar recurrent procedure. Substitution of Eq. (35) into both side of Eq. (34) gives

$$\sum_{i=0}^{\infty} v_i(x,y,t) = \left(-\frac{1}{\rho h}\right)L^{-1}{}_t\left\{(D\,L_{\nabla^4} - n_x\,L_x - n_y\,L_y - 2n_{xy}\,L_{xy})\left[\sum_{i=0}^{\infty} v_i(x,y,t)\right]\right\}$$
$$+\frac{1}{\rho h}L^{-1}{}_t P_z(x,y,t) \tag{36}$$

The use of the decomposition method results in

$$v_0(x,y,t) = \frac{1}{\rho h}L^{-1}{}_t P_z(x,y,t) \tag{37}$$

and for $v_i(x,y,t)$, the recurrent relation becomes

$$v_i(x,y,t)=\left(-\frac{1}{\rho h}\right)L^{-1}{}_t\left[D\,L_{\nabla^4}-n_x\,L_x-n_y\,L_y-2n_{xy}\,L_{xy}\right]v_{i-1}(x,y,t)\quad i\geq 1 \tag{38}$$

For, expanding $v_i(x,y,t)$ in a similar way as used for Eqs. (24) and (25), we obtain

$$\frac{1}{\rho h}P_z(x,y,t)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}N_{kj}\phi_{kj}(x,y) \tag{39}$$

Where the coefficients $N_{kj}(t)$ are provided by the following relation:

$$N_{kj}(t)=\int_{x=0}^{a}\int_{y=0}^{b}\frac{1}{\rho h}P_z(x,y,t)\phi_{kj}(x,y)\,dy\,dx \tag{40}$$

Using Eqs. (32), (37) and (39), we can rewrite Eq. (38) as follows:

$$v_i(x,y,t)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(-1)^i\begin{bmatrix}(\frac{D}{\rho h})^i L^i{}_{\nabla^4}+(-\frac{n_x}{\rho h})^i L^i{}_x\\ +(-\frac{n_y}{\rho h})^i L^i{}_y+(-2\frac{n_{xy}}{\rho h})^i L^i{}_{xy}\end{bmatrix}\phi_{kj}(x,y)\underbrace{(L^{-1}{}_t)^{i+1}N_{kj}(t)}_{I_{i+1}(t)}$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[(-1)^i L^i{}_M\right]\phi_{kj}(x,y)\underbrace{(L^{-1}{}_t)^{i+1}N_{kj}(t)}_{I_{i+1}(t)}\quad i\geq 0$$

$$=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(-1)^i(\lambda_{kj})^i\phi_{kj}(x,y)\underbrace{(L^{-1}{}_t)^{i+1}N_{kj}(t)}_{I_{i+1}(t)}\quad i\geq 0 \tag{41}$$

In order to find $v_i(x,y,t)$, we need to evaluate $(L^{-1}{}_t)^{i+1}N_{kj}(t)$. For this purpose, $I_1(t)$ is written as:

$$I_1(t)=L^{-1}{}_t N_{kj}(t)=\int_0^t\int_0^\tau N_{kj}(\sigma)d\sigma\,d\tau=\int_0^t\int_\sigma^t N_{kj}(\sigma)d\tau\,d\sigma=\int_0^t N_{kj}(\sigma)(t-\sigma)d\sigma \tag{42}$$

And using the same approach for $I_2(t)$ leads to

$$I_2(t)=(L^{-1}{}_t)^2 N_{kj}(t)=L^{-1}{}_t I_1(t)$$

$$=\int_0^t I_1(\sigma)(t-\sigma)d\sigma$$

$$=\int_0^t(t-\sigma)\int_0^\sigma N_{kj}(\tau)(\sigma-\tau)d\tau d\sigma$$

$$=\int_0^t\int_0^\sigma N_{kj}(\tau)(\sigma-\tau)(t-\sigma)d\tau d\sigma$$

$$
\begin{aligned}
&= \int_0^t \int_\tau^t N_{kj}(\tau)(\sigma-\tau)(t-\sigma)\,d\sigma\, d\tau \\
&= \int_0^t N_{kj}(\tau) \int_\tau^t (\sigma-\tau)(t-\sigma)\,d\sigma\, d\tau \\
&= \int_0^t N_{kj}(\tau)\frac{(t-\tau)^3}{3!}\,d\tau = \int_0^t N_{kj}(\sigma)\frac{(t-\sigma)^3}{3!}\,d\sigma
\end{aligned} \tag{43}
$$

Therefore one can write, for $I_{i+1}(t)$,

$$
I_{i+1}(t) = (L^{-1}{}_t)^{i+1} N_{kj}(t) = \int_0^t N_{kj}(\sigma)\frac{(t-\sigma)^{2i+1}}{(2i+1)!}\,d\sigma \tag{44}
$$

Consequently, the final form of the particular solution of the inhomogeneous equation using Eqs. (35), (41) and (44), is as follows:

$$
v_i(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(-1)^i(\lambda_{kj})^i \phi_{kj}(x,y)\underbrace{(L^{-1}{}_t)^{i+1}N_{kj}(t)}_{I_{i+1}(t)} \qquad i \geq 0
$$

$$
v(x,y,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left\{(-1)^i(\lambda_{kj})^i\phi_{kj}(x,y)\left[\int_0^t N_{kj}(\sigma)\frac{(t-\sigma)^{2i+1}}{(2i+1)!}\,d\sigma\right]\right\}
$$

$$
= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\phi_{kj}\left[\int_0^t N_{kj}(\sigma)\left\{\left(\frac{1}{\sqrt{\lambda_{kj}}}\right)\sum_{i=1}^{\infty}(-1)^i\frac{[(t-\sigma)\sqrt{\lambda_{kj}}]^{2i+1}}{(2i+1)!}\right\}d\sigma\right]
$$

$$
v(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[\left(\frac{1}{\sqrt{\lambda_{kj}}}\right)\int_0^t N_{kj}(\sigma)\sin\left((t-\sigma)\sqrt{\lambda_{kj}}\right)d\sigma\right]\phi_{kj} \tag{45}
$$

3.3. General solution of differential equation of lateral motion of thin plate

General solution of the plate equation that is inhomogeneous differential equation is the sum of the solution of the homogeneous problem given by Eq. (33) and solution of the inhomogeneous problem given by Eq. (45).
Therefore the final solution can be written as

$$
w(x,y,t) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[\begin{matrix} a_{kj}\cos(\omega_{kj}t) + \dfrac{b_{kj}}{\omega_{kj}}\sin(\omega_{kj}t) + \\ \left(\dfrac{1}{\omega_{kj}}\right)\displaystyle\int_0^t N_{kj}(\sigma)\sin\left((t-\sigma)\omega_{kj}\right)d\sigma \end{matrix}\right]\phi_{kj}(x,y) \tag{46}
$$

where

$$
\omega_{kj} = (\sqrt{\lambda_{kj}}) \tag{47}
$$

and $k, j = 1, 2, 3, \ldots$ the fundamental mode of flexural vibration is a single sine wave in the X and Y directions, respectively, and ω_{kj} is the kj th mode natural frequency of the plate.

4. Example

The accuracy of the decomposition method is examined by two cases of plate. The results are then compared with known exact solutions.

4.1. Simply supported rectangular plate without lateral and in-plane loadings

As a first case, a rectangular plate is considered with constant D is plate bending stiffness, ρ is the plate material density and h is the plate thickness, which is simply supported at each four edges. The corresponding differential equation of this case is

$$D.\nabla^4 w(x, y, t) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0 \tag{48}$$

With the boundary conditions in this case are [1, 2, 3]

$$w(x = 0, y, t) = 0 \qquad \frac{\partial^2 w(0, y, t)}{\partial x^2} = 0 \tag{49}$$

$$w(x = a, y, t) = 0 \qquad \frac{\partial^2 w(a, y, t)}{\partial x^2} = 0 \tag{50}$$

$$w(x, y = 0, t) = 0 \qquad \frac{\partial^2 w(x, 0, t)}{\partial y^2} = 0 \tag{51}$$

$$w(x, y = b, t) = 0 \qquad \frac{\partial^2 w(x, b, t)}{\partial y^2} = 0 \tag{52}$$

According the Eqs. (29), (30) and (32), the solution in this case, first the generalized Fourier series expansion functions are determined that these functions satisfy the boundary conditions before and after applying the L_M operator. For this case we obtain:

$$L_M = \frac{D}{\rho h} L_{\nabla^4} \tag{53}$$

$$\phi_{kj}(x, y) = \frac{2}{\sqrt{ab}} \sin(\frac{k\pi.x}{a}) \sin(\frac{j\pi.y}{b}) \tag{54}$$

Finally in this case to determined the natural frequency using Eqs. (30), (47), (53) and (54) we obtained:

$$\omega_{kj} = \sqrt{\lambda_{kj}} = \pi^2[(\frac{k}{a})^2 + (\frac{j}{b})^2]\sqrt{\frac{D}{\rho h}} \tag{55}$$

That this solution for natural frequency is exactly the same as the solution of Ref. [1, 2].

4.2. Simply supported plate under in-plane loading

As a second case, the constants and boundary conditions are similar to previous case for a rectangular plate, but in this case we have in-plane loading in parallel to x direction. The corresponding differential equation of this case is

$$D.\nabla^4 w(x,y,t) + \rho h\frac{\partial^2 w(x,y,t)}{\partial t^2} + \frac{n_x}{\rho h}[\frac{\partial^2 w}{\partial x^2}] = 0 \tag{56}$$

In this case, we have a orthogonal function similar to previous case:

$$\phi_{kj}(x,y) = \frac{2}{\sqrt{ab}}\sin(\frac{k\pi.x}{a})\sin(\frac{j\pi.y}{b}) \tag{57}$$

But operator L_M change to this form:

$$L_M = \frac{D}{\rho h}L^i_{\nabla^4} - \frac{n_x}{\rho h}L_x \tag{58}$$

Finally in this case to determined the natural frequency using Eqs. (30), (47), (58) and (57) we obtained:

$$\omega_{kj} = \sqrt{\lambda_{kj}} = \sqrt{\frac{D}{\rho h}\left\{[(\frac{k\pi}{a})^2 + (\frac{j\pi}{b})^2]^2 + \frac{n_x}{D}(\frac{k\pi}{a})^2\right\}} \tag{59}$$

That this solution for natural frequency is exactly the same as the solution of Ref. [1, 2].

5. Conclusions

In this letter Adomian decomposition method has been successfully used to obtain the plates equation. The results obtained by decomposition method are in excellent agreement with classical method. But using the Adomian decomposition method is based upon the orthogonal functions, so developing the method for different applications is not easy and finding these orthogonal functions are also difficult. For example in free end of plate or fixed end of plate finding the orthogonal functions that satisfied the boundary conditions are difficult but this possible.

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