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Solution of Blasius Equation by Decomposition

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Abstract

The Blasius equation is a well-known third-order nonlinear ordinary differential equation, which arises in certain boundary layer problems in the fluid dynamics. In this paper we will construct a decomposition technique defined by

$$u''' = -\frac{1}{2}uu''$$

and a differential operator defined by

$$L = \frac{d}{dx}$$

to obtain a solution as a converging infinite series.

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1. Introduction

The Blasius differential equation arises in the theory of fluid boundary layer mechanics, and in general must be solved numerically as reported in [1], [3], [6] and [7]. In the study of Prandtl boundary layer problems relevant to the motion of an incompressible viscous fluid, solutions of self-similar form naturally give rise to such equations as the Blasius equation. It describes the steady two-dimensional boundary layer that forms on a semi-infinite plate which is held parallel to a constant unidirectional flow u [2]. Concerning the Blasius equation, many researchers have been attempted and much progress has been made so far [4] and [5].

We have been inspired by the recent work of Abbasbandy [1] to come up with a modified decomposition technique to solve the Blasius equation and our solution is consistent with the solution obtained and discussed by Abbasbandy, S [1], Liao, S.J., [5] and Ishimura, Naoyuki [4].

2. The Decomposition Method

Blasius equation $$u''' + \frac{1}{2}uu'' = 0 \quad (1)$$

$$u(x=0) = 0 \quad (1a)$$

$$u'(x=0) = 0 \quad (1b)$$

$$u'(x=\infty) = 1 \quad (1c)$$

We use method of decomposition

$$u = -\frac{1}{2}uu''$$

Define $L = \frac{d}{dx}$, therefore (2)

$$L^3u = -\frac{1}{2}uL^2u$$

or $$u = c_1 + c_2x + c_3x^2 - \frac{1}{2}L^{-3}uL^2u \quad (3)$$

Here $$u_0 = c_1 + c_2 x + c_3 x^2 \tag{4}$$

Applying B.C.'s (1a) and (1b) we get $c_1 = c_2 = 0$

$$\therefore u_0 = c_3 x^2 = cx^2 \tag{5a}$$

and $$L^2 u_0 = 2!cx \tag{5b}$$

$$A_0 = u_0 L^2 u_0 = 2!c^2 x^2 \tag{5c}$$

$$u_1 = -\frac{1}{2} L^{-3} A_0 = -\frac{1}{2} k_1 c^2 \frac{2!x^5}{5!}, \; k_1 = 2! \tag{6a}$$

$$L^2 u_1 = -\frac{1}{2} k_1 c^2 \frac{2!x^3}{3!} \tag{6b}$$

$$\begin{aligned} A_1 &= u_0 L^2 u_1 + u_1 L^2 u_o \\ &= cx^2 - \frac{1}{2} k_1 c^2 \frac{2!x^3}{3!} + -\frac{1}{2} k_1 c^2 \frac{2!x^5}{5!}.2!c \\ &= c^3 . -\frac{1}{2} x^5 \left\{ k_1 \frac{2!}{3!} + k_1 \frac{2!2!}{5!} \right\} \\ &= -\frac{1}{2} c^3 x^5 k_2 \end{aligned} \tag{6c}$$

$$u_2 = -\frac{1}{2} L^{-3} A_1 = \frac{1}{2^2} c^3 k_2 \frac{5!x^8}{8!} \tag{7a}$$

$$L^2 u_2 = \frac{1}{2^2} c^3 k_2 \frac{5!x^6}{6!} \tag{7b}$$

$$A_2 = u_0 L^2 u_2 + u_1 L^2 u_1 + u_2 L^2 u_0$$

$$= cx^2 . \frac{1}{2^2} k_2 c^3 \frac{5!x^6}{6!} - \frac{1}{2} k_1 c^2 \frac{x^5}{5!} - \frac{1}{2} c^2 k_1 \frac{x^3}{3!} + \frac{1}{2^2} k_2 c^3 \frac{5!x^8}{8!} \cdot 2!c$$

$$= \frac{1}{2^2} c^4 x^8 \left\{ k_2 \frac{5!}{6!} + k_1 k_2 \frac{2!}{5!} \frac{2!}{3!} + k_2 \frac{5!}{8!} 2! \right\}$$

$$= \frac{1}{2^2} c^4 x^8 k_3 \tag{7c}$$

$$u_3 = -\frac{1}{2} L^{-3} A_2 = -\frac{1}{2^3} c^4 k_3 \frac{8!\, x^{11}}{11!} \tag{8a}$$

$$L^2 u^3 = -\frac{1}{2^3} c^4 k_3 \frac{8!\, x^9}{9!} \tag{8b}$$

$$A_3 = u_0 L^2 u_3 + u_1 L^2 u_2 + u_2 L^2 u_1 + u_3 L^2 u_0$$

$$= cx^2 \cdot (-\frac{1}{2^3} k_3 c^4 \frac{8!\, x^9}{9!}) + (-\frac{1}{2} k_1 c^2 \frac{2!\, x^5}{5!} \cdot \frac{1}{2^2} k_2 c^3 \frac{5!\, x^6}{6!}) - \frac{1}{2^2} k_2 c^3 \frac{5!\, x^8}{8!} \cdot \frac{1}{2} c^2 k_1 \frac{2!\, x^3}{3!} - \frac{1}{2^3} k_3 c^4 \frac{8!\, x^{11}}{11!} 2!c$$

$$= -\frac{1}{2^3} c^5 x^{11} \left\{ k_3 \frac{8!}{9!} + k_1 k_2 \frac{2!}{5!} \cdot \frac{5!}{6!} + k_1 k_2 \frac{5!}{8!} \cdot \frac{2!}{3!} + k_3 \frac{8!}{11!} .2! \right\}$$

$$= -\frac{1}{2^2} c^5 x^{11} k^4 \tag{8c}$$

$$u_4 = -\frac{1}{2} L^{-3} A_3 = \frac{1}{2^4} c^5 k_4 \frac{11!\, x^{14}}{14!} \tag{9a}$$

$$L^2 u_4 = \frac{1}{2^4} c^5 k_4 \frac{11!\, x^{12}}{12!} \tag{9b}$$

By deduction we get A_4

$$A_4 = u_0 L^2 u_4 + u_1 L^2 u_3 + u_2 L^2 u_2 + u_3 L^2 u_1 + u_4 L^2 u_0$$

$$= \frac{1}{2^4} c^5 x^{14} \left\{ k_4 \frac{11!}{12!} + k_1 k_3 \frac{2!}{5!} \cdot \frac{8!}{9!} + k_1 k_2 \frac{5!}{8!} \cdot \frac{5!}{6!} + k_3 k_1 \frac{8!}{11!} . \frac{2!}{3!} + k_4 \frac{11!2!}{14!} \right\}$$

$$= \frac{1}{2^4} c^6 x^{14} k_5 \tag{9c}$$

In general,

$$k_{n+1} = k_n \left\{ \frac{1}{3^n} + \frac{2}{3n(3n+1)(3n+2)} \right\} + \sum_{i=1}^{n-1} k_i k_{n-i} \frac{(3i-1)![3(n-i)-1]!}{(3i+2)![3(n-i)]!}; \; n \geq 2 \quad (10)$$

After simplification we get:

$$\mathrm{k}_{\mathrm{n+1}} = \mathrm{k}_{\mathrm{n}} \frac{(3n+1)(3n+2)+2}{3n(3n+1)(3n+2)} + \frac{1}{9} \sum_{i=1}^{n-1} \frac{k_i k_{n-i}}{i(3i+1)(3i+2)(n-i)} \quad (11)$$

and the solution u_n is given by,

$$u_n = \left(-\frac{1}{2} \right)^n k_n c^{n+1} \frac{(3n+1)!\, x^{3n+2}}{(3n+2)!} \quad n \geq 1 \quad (12)$$

Here $k_0 = 1, k_1 = 2, k_2 = \frac{1}{3} + \frac{2}{3.4.5} = \frac{22}{3.4.5} = \frac{11}{30}$ (13)

$$u_0 = cx^2 \quad (14)$$

and $c = \frac{1}{6}$ (15)

the total solution, $u = \sum_{n=0}^{\infty} u_n$ (16)

In summary:

For the Blasius equation: $\frac{d^3u}{dx^3} + \frac{1}{2} u \frac{d^2u}{dx^2} = 0$

The complete solution is given by,

$$u = \sum_{n=0}^{\infty} u_n$$

where $u_0 = cx^2$ with $c = \frac{1}{6}$

and $$u_n = \left(-\frac{1}{2}\right)^n k_n c^{n+1} \frac{x^{3n+2}}{3n(3n+1)(3n+2)}; \quad n \geq 1$$

and

$$\mathrm{k}_{\mathrm{n+1}} = \mathrm{k}_{\mathrm{n}} \frac{(3n+1)(3n+2)+2}{3n(3n+1)(3n+2)} + \frac{1}{9}\sum_{i=1}^{n-1} \frac{k_i k_{n-i}}{i(3i+1)(3i+2)(n-i)}; n \geq 2$$

with $k_0 = 1$, $k_1 = 2$ and $k_2 = \frac{11}{30}$

The velocity field is $u' = \sum_{n=0}^{\infty} u'_n$

$$u'_n = \left(-\frac{1}{2}\right)^n k_n c^{n+1} \frac{x^{3n+1}}{3n(3n+1)}$$

Remark: The solution obtained above is consistent with the solutions given in [1] and [3] both numerically and analytically.

References

[1] S. Abbasbandy, A Numerical Solution of Blasius Equation by Adomian's Decomposition Method and Comparison with Homotopy Perturbation Method, Chaos, Solitons and Fractals, 31 (2007), 257-260.

[2] Z. Belhachmi, B. Brighi, and Taous, On The Concave Solutions of the Blasius Equation, Acta Math. Univ. Comenianae, LXIX (2000), 199-214.

[3] H. Ji-Huan, A simple Perturbation Approach to Blasius Equation, Applied Mathematics and Computation, 140 Issues 2-3 (2003), 217-222.

[4] S. J. Liao, An Explicit, Totally Analytical Approximate Solution for Blasius Viscous Flow Problem, International Journal of Non-Linear Mechanics, 34 (1999), 759-778.

[5] I. Naoyuki., On Blowing-Up Solutions of the Blasius Equation, Discrete and Continuous Dynamical Systems, 9 Number 4 (2003).

[6] C. Pozrikidis, Introduction to Theoritical and Computational Fluid Dynamics, Oxford, 1998.

[7] H. Schlichting, Boundary-Layer Theory, Springer, New York, 2004.

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