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Explicit Non-Algebraic Limit Cycle for Polynomial Systems of Degree Seven

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Abstract

In this work, we determine conditions for planar systems of the form

$$\begin{cases} \frac{dx}{dt} = P_5(x,y) + xR_6(x,y) \\ \frac{dy}{dt} = Q_5(x,y) + yR_6(x,y) \end{cases} \quad S(a,b,c,u,v,w)$$

where

$P_5(x,y) = ax^5 + bx^4y + cx^3y^2 + ux^2y^3 + vxy^4 + wy^5$,

$Q_5(x,y) = -6wx^5 + ax^4y + (b-11w)x^3y^2 + cx^2y^3 + (u-6w)xy^4 + vy^5$,

$R_6(x,y) = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$,

and where a, b, c, u, v and w are real constants, to possess non-algebraic limit cycles. Moreover we proof that this non-algebraic limit cycle, when it exists it can be explicitly given.

This is done as an application of former theorems gives description of the existence of the non-algebraic limit cycles of the family of systems:

$$\begin{cases} \frac{dx}{dt} = P_n(x,y) + xR_n(x,y), \\ \frac{dy}{dt} = Q_n(x,y) + yR_n(x,y), \end{cases}$$

where $P_n(x,y), Q_n(x,y)$ and $R_n(x,y)$ are homogenous polynomials of degree n, n and m respectively with $n < m$ and n is odd, m is even. The tool for proving our result is based on methods developed in [1] and [2].

Keywords: Planar polynomial differential system, Algebraic solution, non-algebraic limit cycle

Mathematics Subject Classification: 34A05, 34C07

1 Introduction

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of the form

$$\begin{cases} \frac{dx}{dt} = X(x,y) \\ \frac{dy}{dt} = Y(x,y) \end{cases} \tag{1}$$

where $X(x,y)$ and $Y(x,y)$ are coprime polynomials. A limit cycle of system (1) is an isolated periodic orbit and it is said to be algebraic if it is contained in the zero set of an algebraic curve.

We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem. And an even more difficult problem is to give an explicit expression of them. We are able to solve this last problem for a given system of the form (1). Until recently, the only limit cycles known in an explicit way were algebraic. In [1],[2] and [3] examples of explicit limit cycles which are not algebraic are given. Limit cycles of planar polynomial differential systems are not, in general, algebraic. For instance, the limit cycle appearing in van der Pol's system is not algebraic as it is proved in [4].

In this work, we shall be essentially concerned with a planar polynomial system of the form

$$\begin{cases} \frac{dx}{dt} = P_n(x,y) + xR_m(x,y), \\ \frac{dy}{dt} = Q_n(x,y) + yR_m(x,y), \end{cases} \tag{2}$$

where $P_n(x,y), Q_n(x,y)$ and $R_n(x,y)$ are homogenous polynomials of degree n, n and m respectively with $n < m$ and n is odd, m is even.

it is a fact that system (2) has at most one limit cycle. system (2) has, from hypothesis of P_n and Q_n and using the Euler formula, that $F(x,y) = yP_n(x,y) - xQ(x,y)$ as an algebraic solution with cofactor $(n+1)R_m + \frac{\partial P_n}{\partial x} + \frac{\partial Q_n}{\partial y}$. Notice that this algebraic solution is formed by a product (complex or real) invariant straight lines though the origin. Moreover if $F(x,y)$ is an invariant algebraic solution of degree l of the system (2), then the homogeneous part of maximum degree of its cofactor is $lR_m(x,y)$, for more details, see [1]and [2]. We apply these properties to determine conditions of the parameters a, b, c, u, v and w for the planar systems

$$\begin{cases} \frac{dx}{dt} = P_5(x,y) + xR_6(x,y) \\ \frac{dy}{dt} = Q_5(x,y) + yR_6(x,y) \end{cases} \quad S(a,b,c,u,v,w)$$

where

$$\begin{aligned} P_5(x,y) &= ax^5 + bx^4y + cx^3y^2 + ux^2y^3 + vxy^4 + wy^5, \\ Q_5(x,y) &= -6wx^5 + ax^4y + (b-11w)x^3y^2 + cx^2y^3 + (u-6w)xy^4 + vy^5, \\ R_6(x,y) &= x^6 + 3x^4y^2 + 3x^2y^4 + y^6, \end{aligned}$$

in order to possess non-algebraic limit cycles. The existence of non-algebraic limit cycle of a such system with P_3, Q_3 and R_4 has been studied in [1] and[2].

To prepare the demonstration of our result, one needs some preliminaries.

Lemma 1 . *Assume that system (2) has an algebraic solution $F(x,y)$, then the corresponding cofactor $K(x,y)$ is in the form*

$$K(x,y) = \left[\sum_{j=0}^{\frac{m-2}{2}} K_{2j}(x,y)\right] + lR_m(x,y) \tag{3}$$

where $K_{2j}(x,y)$ is homogeneous polynomial of degree $2j$ and l is the degree of F.

for the proof see ([1], Lemma 1).

Definition 1 . *For system (2), real or complex numbers α_k for which*
$yP_n(x,y) - xQ_n(x,y) = \prod_{k=1}^{n+1}(y - \alpha_k x)$
are called invariant slopes of the system.

The following lemma concerns with system (2), giving condition of the algebraiticity of the limit cycles, if exists (see [1]).

Lemma 2 . *For $l \neq 0$, if the expression*

$$C_0 \exp \int \frac{\left[\sum_{j=0}^{\frac{m-2}{2}} K_{2j}(1,\alpha_k)s^{2j}\right] + lR_m(1,\alpha_k)s^m}{P_n(1,\alpha_k)s^n + R_m(1,\alpha_k)s^{m+1}} ds \tag{4}$$

for all invariant slopes α_k, are not polynomials, then the limit cycle, if exists, is non-algebraic.

For the proof see [1, Theorem 1] and [2, Lemma 2.4].

2 Main result

We proof the following result:

Theorem 3 . *If* $a<0,(a-c+v)>0,(c-2v)>0,$*and* $w>0$*, then the planar differential system*

$$\begin{cases} \frac{dx}{dt}=P_5(x,y)+xR_6(x,y) \\ \frac{dy}{dt}=Q_5(x,y)+yR_6(x,y) \end{cases} S(a,b,c,u,v,w)$$

where

$$\begin{aligned} P_5(x,y) &= ax^5+bx^4y+cx^3y^2+ux^2y^3+vxy^4+wy^5, \\ Q_5(x,y) &= -6wx^5+ax^4y+(b-11w)x^3y^2+cx^2y^3+(u-6w)xy^4+vy^5, \\ R_6(x,y) &= x^6+3x^4y^2+3x^2y^4+y^6, \end{aligned}$$

has exactly one limit cycle.

Proof.

System $S(a,b,c,u,v,w)$ can be written in polar coordinates (r,θ) defined by $x=r\cos\theta,\ y=r\sin\theta$, as

$$\begin{cases} r'=\sigma(\theta)r^5+r^7 \\ \theta'=-\left(2\cos^4\theta+3\cos^2\theta+1\right)wr^4 \end{cases} \tag{5}$$

where

$$\sigma(\theta)=(a-c+v)\cos^4\theta+(c-2v)\cos^2\theta \\ +(\cos\theta\sin\theta)\left((b-w-u)\cos^2\theta+u-5w\right)+v$$

System (5) can be written as

$$\begin{aligned} \frac{dr}{d\theta}=-&\frac{\left((a-c+v)\left(\cos^4\theta\right)+(c-2v)\left(\cos^2\theta\right)+(\cos\theta\sin\theta)\left(\left(\cos^2\theta\right)(b-w-u)+u-5w\right)+v\right)}{\left(2\cos^4\theta+3\cos^2\theta+1\right)w}r \\ &-\frac{1}{\left(2\cos^4\theta+3\cos^2\theta+1\right)w}r^3 \end{aligned} \tag{6}$$

which is a Bernoulli equation. By introducing the change of variables $\rho=\frac{1}{r^2}$, we obtain the linear differential equation

$$\frac{d\rho}{d\theta}=f(\theta)\rho+g(\theta)$$

where

$f(\theta)=2\frac{\left((a-c+v)\left(\cos^4\theta\right)+(c-2v)\left(\cos^2\theta\right)+(\cos\theta\sin\theta)\left(\left(\cos^2\theta\right)(b-w-u)+u-5w\right)+v\right)}{\left(2\cos^4\theta+3\cos^2\theta+1\right)w}$,

$g(\theta)=2\frac{1}{\left(2\cos^4\theta+3\cos^2\theta+1\right)w}$.

Notice that system $S(a,b,c,u,v,w)$ has a periodic solution if and only if equation (6) has a strictly positive 2π-periodic solution. The solution satisfying the initial condition $\rho(0)=\rho_0>0$ is given by

$$\rho(\theta,\rho_0)=\left[\exp\left(\int_0^\theta f(s)\,ds\right)\right]\left[\rho_0+\int_0^\theta\left(g(s)\exp\left(-\int_0^s f(\xi)\,d\xi\right)\right)ds\right]$$
$$=$$
$$\exp\left(\left(\int_0^\theta f(s)\,ds\right)\right)\left[\rho_0+\int_0^\theta\left(2\frac{(a-b+c)\cos^4 s+(b-2c)\cos^2 s+c}{(1+4\lambda)(\cos^2 s+1)}\exp\left(\left(-\int_0^s f(\xi)\,d\xi\right)\right)\right)ds\right]$$

The condition of the periodic solution of period 2π starting at $\rho=\rho_0>0$ is given by the equation $\rho(2\pi)=\rho(0)$. This implies

$$\rho_0=\frac{\exp\left(\int_0^{2\pi} f(s)ds\right)}{1-\exp\left(\int_0^{2\pi} f(s)ds\right)}\int_0^{2\pi}\left(g(s)\exp\left(-\int_0^s f(\xi)\,d\xi\right)\right)ds,$$

recall that $\rho_0=r_0^{-2}$ where $r_0>0$ is the intersection of periodic solution with the positive $x-$axis. So the existence of such r_0, and consequently the existence of the periodic solution, needs ρ_0 to be strictly positive.

We have

$$\int_0^{2\pi} f(s)\,ds=$$
$$\int_0^{2\pi}\left(2\frac{\left((a-c+v)\left(\cos^4\theta\right)+(c-2v)\left(\cos^2\theta\right)+(\cos\theta\sin\theta)\left(\left(\cos^2\theta\right)(b-w-u)+u-5w\right)+v\right)}{(2\cos^4\theta+3\cos^2\theta+1)w}\right)d\theta=$$
$$\int_0^{2\pi}\left(2\frac{(a-c+v)\left(\cos^4\theta\right)+(c-2v)\left(\cos^2\theta\right)+v}{(2\cos^4\theta+3\cos^2\theta+1)w}\right)d\theta,$$

or $a<0,(a-c+v)>0,(c-2v)>0$,and $w>0$ so

$$\int_0^{2\pi} f(s)\,ds<\int_0^{2\pi}\left(2\frac{a}{(2\cos^4\theta+3\cos^2\theta+1)w}\right)d\theta<0.$$

Consequently, we have

$$0<\exp\left(\int_0^{2\pi} f(s)\,ds\right)<1,$$

and

$$\frac{\exp\left(\int_0^{2\pi} f(s)ds\right)}{1-\exp\left(\int_0^{2\pi} f(s)ds\right)}>0.$$

On the other hand, we have from hypothesis of the theorem that $w>0$ so

$$\int_0^{2\pi}\left(g(\theta)\exp\left(-\int_0^\theta f(s)\,ds\right)\right)d\theta$$
$$=\int_0^{2\pi}\left(2\frac{1}{(2\cos^4\theta+3\cos^2\theta+1)w}\exp\left(-\int_0^\theta f(s)\,ds\right)\right)d\theta>0,$$

and by consequence one has

$$\rho_0=\frac{\exp\left(\int_0^{2\pi} f(s)ds\right)}{1-\exp\left(\int_0^{2\pi} f(s)ds\right)}\int_0^{2\pi}\left(g(s)\exp\left(-\int_0^s f(\xi)\,d\xi\right)\right)ds>0$$

Hence the strictly positive 2π-periodic solution of equation (6) does exist.

In order to prove that this periodic solution is an isolated periodic orbit,see for instance [5], it is sufficient to that the poincare return map $\prod(\rho_0) = \rho(2\pi, u_0)$, has $\frac{d\prod}{du_0}(\rho_0) \neq 1$ for all ρ_0 and this is already satisfied, because

$$\frac{d\prod}{d\rho_0}(\rho_0) = \exp\left(\int_0^{2\pi} f(s)\, ds\right) < 1 \text{ for all } \rho_0 \blacksquare$$

Even if the explicit expression of the limit cycle of system $S(a, b, c, u, v, w)$ is found, is not an easy task to decide whether this curve is algebraic or not unless we make other investigation.

3 Algebraic solution

Recall (see [2]) that a real or complex polynomial $F(x, y)$ is an algebraic solution of a real polynomial system

$$\begin{cases} \frac{dx}{dt} = X(x, y) \\ \frac{dy}{dt} = Y(x, y) \end{cases}$$

if

$$X(x, y)\frac{\partial F}{\partial y}(x, y) + Y(x, y)\frac{\partial F}{\partial y}(x, y) = K(x, y)F(x, y)$$

for some polynomial $K(x,y)$, called the cofactor of $F(x, y)$. Notice that when $F(x,y)$ is real, the curve $F(x,y) = 0$ is invariant under the flow of the differential system. Observe also that the degree of the cofactor is one less than the degree of the vector field. A limit cycle is called algebraic if it is an oval of a real algebraic solution.

It is not difficult to see that system $S(a, b, c, u, v, w)$ has an algebraic solution.

Lemma 4 *System $S(a, b, c, u, v, w)$ has*

$F(x, y) = (x^2 + y^2)(2x^2 + y^2)(3x^2 + y^2)$ as an algebraic solution with cofactor $K(x, y) = 6ax^4 - 22x^3wy + 6x^3by + 6cy^2x^2 - 24xwy^3 + 6xuy^3 + 6vy^4 + 6(x^6 + 3x^4y^2 + 3x^2y^4 + y^6)$

Proof. . If we put

$$P_5(x, y) = (ax^5 + bx^4y + cx^3y^2 + ux^2y^3 + vxy^4 + wy^5),$$
$$Q_5(x, y) = (-6wx^5 + ax^4y + (b - 11w)x^3y^2 + cx^2y^3 + (u - 6w)xy^4 + vy^5)$$
$$R_6(x, y) = (x^6 + 3x^4y^2 + 3x^2y^4 + y^6)$$

immediately we have

$$
\begin{gathered}
(P_5(x,y) + xR_6(x,y)) \tfrac{d}{dx}\left(\left(x^2+y^2\right)\left(2x^2+y^2\right)\left(3x^2+y^2\right)\right) \\
+ (Q_5(x,y) + yR_6(x,y)) \tfrac{d}{dy}\left(\left(x^2+y^2\right)\left(2x^2+y^2\right)\left(3x^2+y^2\right)\right) = \\
\left(\left(x^2+y^2\right)\left(2x^2+y^2\right)\left(3x^2+y^2\right)\right) K(x,y),
\end{gathered}
$$

it is what one wants to show■.

4 A non-algebraic limit cycles

This section is devoted to prove that the limit cycles of the parametrized systems $S(a,b,c,\lambda)$ is not algebraic. Indeed, we will prove that the only algebraic solutions of the system $S(a,b,c,\lambda)$ are the ones given in the above Lemma.

Lemma 5 . *If* $(a-2c+4v)(b-2u+4w) \neq 0$, *then system* $S(a,b,c,u,v,w)$ *has only three algebraic solutions* $x^2+y^2=0, 2x^2+y^2=0$ *and* $3x^2+y^2=0$.

Proof. . These three curves coincide with the six complex lines $y = (\pm)\, ix, y = (\pm)\sqrt{2}ix$ and $y = (\pm)\sqrt{3}ix$. We apply Lemma 3 .

Assume that the differential system $S(a,b,c,u,v,w)$ has a real or complex algebraic solution $F(x,y)$ and that it does not contain any of the given six lines as a factor. By using (Lemma 2.6 [2]) it is not restrictive to assume that $F(x,y)$ is real and that its cofactor is an even function, i.e., $K(-x,-y) = K(x,y)$.

Since the degree of the system $S(a,b,c,u,v,w)$ is 7 we know that the degree of $K(x,y)$ is at most 6. By the above restrictions on $K(x,y)$ and by using also (Lemma 2.5 [2]) we can write it as the real polynomial

$$
\begin{gathered}
K(x,y) = d_0 + d_1x^2 + d_2xy + d_3y^2 + b_4x^4 + b_3x^3y + b_2x^2y^2 + b_1xy^3 + b_0y^4 + \\
l\left(x^6 + 3x^4y^2 + 3x^2y^4 + y^6\right)
\end{gathered}
$$

where l is the degree of the corresponding algebraic curve $F(x,y) = 0$. By considering the invariant slopes $\alpha_{3,4} = (\pm)\sqrt{2}i$,and recalling that for system $S(a,b,c,u,v,w)$ we have $m = 7, n = 5$ the expression (4) will be

$$
C_0 \exp \int^x \frac{\left[\sum_{j=0}^{2} K_{2j}\left(1,(\pm)\sqrt{2}i\right)s^{2j}\right] + lR_6\left(1,(\pm)\sqrt{2}i\right)s^6}{P_5\left(1,(\pm)\sqrt{2}i\right)s^5 + R_6\left(1,(\pm)\sqrt{2}i\right)s^6} ds = C_0 \exp \int^x \left(\frac{ls^6+\gamma s^4+\delta s^2-d_0}{\left(s^2+\lambda+i\beta\right)s^5}\right) ds
$$

where

$\lambda = (-a+2c-4v)$

$\beta = -\sqrt{2}\,(b-2u+4w)$

$\gamma = \left(-b_4 - ib_3\sqrt{2} + 2b_2 + 2ib_1\sqrt{2} - 4b_0\right)$

$\delta = \left(-d_1 - id_2\sqrt{2} + 2d_3\right)$

$$C_0 \exp \int^x \left(\frac{ls^6+\gamma s^4+\delta s^2-d_0}{(s^2+\lambda+i\beta)s^5}\right) ds = C_0 \exp \int^x \left(\frac{(l-B)s}{(s^2+\lambda+i\beta)} + \frac{B}{s} + \frac{B_3}{s^3} - \frac{d_0}{(\lambda+i\beta)s^5}\right) ds$$

$$= C_0 \exp \left(\begin{array}{c} (l-B)\int^x \left(-\frac{1}{2}i\frac{\frac{2s}{\beta}}{\left(\frac{s^2+\lambda}{\beta}\right)^2+1} + \frac{1}{4}\frac{4s\left(s^2+\lambda\right)}{(s^2+\lambda)^2+\beta^2}\right) ds \\ +\int^x \left(\frac{B}{s} + \frac{B_3}{s^3} + \frac{-\frac{1}{\lambda+i\beta}d_0}{s^5}\right) ds \end{array} \right)$$

where
$B = -\frac{\delta\lambda+i\delta\beta-2i\beta\gamma\lambda+\beta^2\gamma-\lambda^2\gamma+d_0}{\lambda^3+3i\lambda^2\beta-3\beta^2\lambda-i\beta^3}$
$B_3 = \frac{d_0+\delta\lambda+i\delta\beta}{\lambda^2+2i\lambda\beta-\beta^2}$
then

$$C_0 \exp \int^x \left(\frac{ls^6+\gamma s^4+\delta s^2-d_0}{(s^2+\lambda+i\beta)s^5}\right) ds =$$

$$C_0 \exp \left(\begin{array}{c} (l-B)\left(-\frac{1}{2}i\arctan\left(\left(\frac{x^2+\lambda}{\beta}\right)\right) + \frac{1}{4}\ln\left(\left(x^2+\lambda\right)^2+\beta^2\right)\right) \\ +B\ln x - \frac{B_3}{2x^2} + \frac{d_0}{4(\lambda+i\beta)x^4} \end{array} \right),$$

now by forcing this expression to be a real polynomial with C_0 an arbitrary constant, we have among others conditions a first set of necessary conditions:

$$\begin{cases} d_0 = 0 \\ d_0 + \delta\lambda + i\delta\beta = 0 \\ l\lambda^3 - 3l\beta^2\lambda + \delta\lambda + \beta^2\gamma - \lambda^2\gamma = 0 \\ 3l\lambda^2\beta - l\beta^3 + \delta\beta - 2\beta\gamma\lambda = 0. \end{cases}$$

One recalls that $\lambda \neq 0$ and $\beta \neq 0$, we get that $l = 0$. This implies that there is no algebraic solution except $x^2+y^2 = 0, 2x^2+y^2 = 0$,and $3x^2+y^2 = 0$ for the system $S(a,b,c,u,v,w)$ ■.

Theorem 6 . *If $a < 0, (a-c+v) > 0, (c-2v) > 0,$ and $w > 0$, system $S(a,b,c,u,v,w)$ has exactly one limit cycle.*

This limit cycle is non-algebraic if $(a-2c+4v)(b-2u+4w) \neq 0$.

Proof. . The proof of the theorem follows immediately from the results of Theorem 4 and Lemma 6.■

Remark 1 . *In polar coordinates the expression of the limit cycle of system $S(a,b,c,u,v,w)$ is*

$$\frac{1}{r^2} = \left[\begin{array}{c} \int_0^\theta \left(2\frac{(a-b+c)\cos^4 s+(b-2c)\cos^2 s+c}{(1+4\lambda)(\cos^2 s+1)} \exp\left(-\int_0^s f(\xi)\,d\xi\right)\right) ds \\ +\frac{1}{r_0^2} \end{array} \right] \exp\left(\int_0^\theta f(s)\,ds\right)$$

where

$f(\theta) = 2\frac{\left((a-c+v)\left(\cos^4\theta\right)+(c-2v)\left(\cos^2\theta\right)+(\cos\theta\sin\theta)\left(\left(\cos^2\theta\right)(b-w-u)+u-5w\right)+v\right)}{(2\cos^4\theta+3\cos^2\theta+1)w},$

$g(\theta) = 2\frac{1}{(2\cos^4\theta+3\cos^2\theta+1)w}.$

$\frac{1}{r_0^2} = \frac{\exp\left(\int_0^{2\pi} f(s)ds\right)}{1-\exp\left(\int_0^{2\pi} f(s)ds\right)} \int_0^{2\pi} \left(g(s)\exp\left(-\int_0^s f(\xi)\,d\xi\right)\right) ds,$

Remark 2 *and it is not an easy task to elucidate this expression in cartesian coordinates.*

5 Examples

The following example is given to Illustrate our result.

Example 1 . *Let's consider the system*

$$\begin{cases} \frac{dx}{dt} = (-x^5 - 6x^3y^2 + x^2y^3 - 4xy^4 + y^5) + xR_6(x,y) \\ \frac{dy}{dt} = (-6x^5 - x^4y - 11x^3y^2 - 6x^2y^3 - 5xy^4 - 4y^5) + yR_6(x,y) \end{cases}$$

This system is of the form $S(-1,0,-6,1,-4,1)$. *We have*

$a = -1 < 0, (a-c+v) = 1 > 0, (c-2v) = 2 > 0,$ *and* $w = 1 > 0.$

So the first hypothesis of Theorem 7 is satisfied and hence the system has exactly one limit cycle.

The second hypothesis is satisfied as well, $(a-2c+4v)(b-2u+4w) = -10 \neq 0$. *Thus this limit cycle is non-algebraic.*

Now we consider an other example.

Example 2 . *Let's consider the system*

$(b-2u+4w)$, *Solution is :* $b = 2u - 4w = 2 - 4 = -2$

$$\begin{cases} \frac{dx}{dt} = (-x^5 - 2x^4y - 6x^3y^2 + x^2y^3 - 4xy^4 + y^5) + xR_6(x,y) \\ \frac{dy}{dt} = (-6x^5 - x^4y - 13x^3y^2 - 6x^2y^3 - 5xy^4 + vy^5) + yR_6(x,y) \end{cases}$$

This system is of the form $S(-1,-2,-6,1,-4,1)$. *We have*

$a = -1 < 0, (a-c+v) = 1 > 0, (c-2v) = 2 > 0,$ *and* $w = 1 > 0.$

So the first hypothesis of Theorem 7 is satisfied and hence the system has exactly one limit cycle.

The second condition is not satisfied, because $(b-2u+4w) = 0$, *it is why one cannot say anything concerning the algebraiticity of this limit cycle.*

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