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Modified Neumann Series for Solving Fredholm Integral Equation

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Abstract

In this paper we present a modification to Neumann series method for solving Fredholm integral equation. We create a perturbation in left side of Fredholm integral equation and we show the decrement of the upper bound for the error.

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1 Introduction

We are frequently faced with the problem of determining the solution of integral equations, one of these integral equations is the Fredholm integral equation where defined in [4, 5, 6]. In [4, 6] used of the Neumann series for solve of Fredholm integral equation. In this paper we use of the Perturbation Method [7, 8] in Neumann series and we will show the upper bound for the error will be decreased.

In this paper we use some preliminaries in Section 2, and the main idea and the main result of our work are introduced in Section 3. Some examples are presented in Section 4.

2 Preliminary Notes

Consider the Fredholm integral equation:

$$\phi(x) = f(x) + \int_a^b K(x,y)\phi(y)dy, \qquad a \leq x \leq b. \tag{1}$$

Definition 2.1 *If for a given $\lambda = \lambda_0$ the inverse operator L^{-1} exists such that $LL^{-1} = I = L^{-1}L$ holds that $L = I - \lambda K$, then λ_0 is known as a regular value of the operator K.*

If λ is a regular value of K, then (1) has the unique solution

$$\phi = (I - \lambda K)^{-1} f = L^{-1} f. \tag{2}$$

Theorem 2.2 *If $||\lambda K|| < 1$ then [4, 6]*

$$L^{-1} = (I - \lambda K)^{-1} = I + \sum_{n=1}^{\infty} \lambda^n K^n,$$

and an upper bound for the error is as follows

$$||e_n|| \leq \frac{||\phi_{n+1} - \phi_n||}{1 - ||\lambda K||}. \tag{3}$$

3 Main Results

In this section we add "$\varepsilon\phi$" to the left side of (1) and prove that the upper bound for the error will be decreased. First we can write (1) into form

$$\phi = f + \lambda K \phi, \tag{4}$$

now we add "$\varepsilon\phi$" to the left side of (4) and we have

$$\phi + \varepsilon\phi = f + \lambda K \phi. \tag{5}$$

If $||\lambda K|| < 1$ then from the definition of Neumann series [4, 6] we have

$$\phi = \sum_{i=0}^{\infty} \lambda^i K^i f \ .$$

If we truncate the series $\phi = \sum_{i=0}^{\infty} \lambda^i K^i f$ after n terms, and denote

$$\phi_n = \sum_{i=0}^{n} \lambda^i K^i f \ , \tag{6}$$

then we use the recurrence relations

$$\phi_0 = f \ ,$$
$$\phi_{n+1} = f + \lambda K \phi_n \ , \tag{7}$$

where

$$K\phi_n = \int_a^b K(x,y)\phi_n(y)dy. \tag{8}$$

Theorem 3.1 *If we add "$\varepsilon\phi$" to the left side of (1) then for the upper bound of the error we have*

$$||e_n|| \leq \frac{||\phi_{n+1} - \phi_n||}{1 - \frac{||\lambda K||}{|1+\varepsilon|}} \ , \tag{9}$$

we know $\{\forall \varepsilon > 0 : |1+\varepsilon| = 1 + |\varepsilon|\}$ *thereupon for (9) we have*

$$||e_n|| \leq \frac{||\phi_{n+1} - \phi_n||}{1 - \frac{||\lambda K||}{1+|\varepsilon|}} \ . \tag{10}$$

Proof 3.2 *By adding "$\varepsilon\phi_{n+1}$" to the left side of (7) we have*

$$\phi_{n+1} + \varepsilon\phi_{n+1} = f + \lambda K \phi_n \ . \tag{11}$$

We define $e_n = \phi - \phi_n$ *and using (5) and (11) we have*

$$(1+\varepsilon)\phi = f + \lambda K\phi,$$
$$(1+\varepsilon)\phi_{n+1} = f + \lambda K\phi_n,$$

then

$$(1+\varepsilon)(\phi - \phi_{n+1}) = \lambda K(\phi - \phi_n)$$

however

$$(1+\varepsilon)e_{n+1} = \lambda K e_n \quad ; e_n = (\phi - \phi_n)$$

since

$$||(1+\varepsilon)e_{n+1}|| = ||\lambda K e_n||$$

then we have

$$||e_{n+1}|| \leq \frac{||\lambda K||}{|1+\varepsilon|}||e_n|| \ . \tag{12}$$

Moreover

$$||e_n|| \leq ||e_{n+1}|| + ||\phi_{n+1} - \phi_n||,$$

then

$$||e_n|| \leq \frac{||\phi_{n+1} - \phi_n||}{1 - \frac{||\lambda \ K||}{|1+\varepsilon|}} \ ,$$

then we have

$$||e_n|| \leq \frac{||\phi_{n+1} - \phi_n||}{1 - \frac{||\lambda K||}{1+|\varepsilon|}} \ .$$

Lemma 3.3 *If $|\varepsilon| \leq ||\lambda K||$ then the upper bound for the error will be decreased.*

Proof 3.4 *Since*

$$\frac{||\lambda K||}{1+|\varepsilon|} \leq ||\lambda K||,$$

then we have

$$1 - ||\lambda K|| \leq 1 - \frac{||\lambda K||}{1+|\varepsilon|},$$

then

$$\frac{||\phi_{n+1} - \phi_n||}{1 - \frac{||\lambda K||}{1+|\varepsilon|}} \leq \frac{||\phi_{n+1} - \phi_n||}{1 - ||\lambda K||}, \tag{13}$$

now by applying (3) and (10) and compare between them by the (13) claim is proved, namely upper bound for the error is decreased.

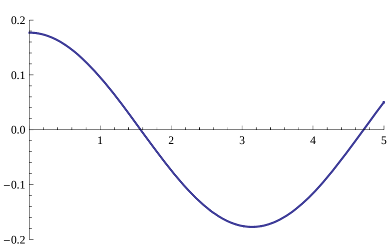
4 Numerical Example

We used two methods, Neumann series and our method (Modified Neumann Series), to solve 3 integral equations in examples 1 and 2 and 3 we compare the results.

Example 4.1 *Consider the following Fredholm integral equation*

$$\phi(x) = 5\cos x - \frac{1}{4}\cos x\ \sin^2(1) + 0.1\int_0^1 \sin y \cos x\ \phi(y)dy$$

Exact solution is $\phi(x) = 5\cos x$, and via $\varepsilon = 0.032$, the graphs of the errors for x_{10} in both methods are as follows:



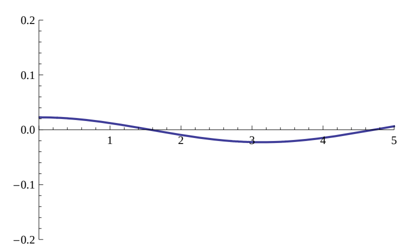
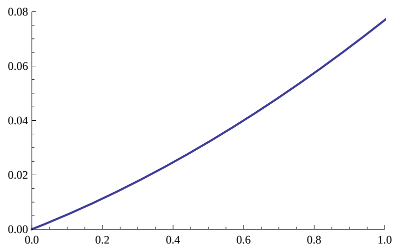


Figure 1 : Error of Neumann series   Figure 2 : Modified Neumann series ($\varepsilon = 0.032$).

Example 4.2 *Consider the following Fredholm integral equation*

$$\phi(x) = x^3 + 2x + \frac{1}{20}\left(-\frac{26}{15}\,x - \frac{13}{15}\,x^2\right) + 0.05\int_0^1 y\left(x^2 + 2x\right)\phi(y)dy$$

Exact solution is $\phi(x) = x^3 + 2x$, *and via* $\varepsilon = 0.017$, *the graphs of the errors for* x_{10} *in both methods are as follows:*



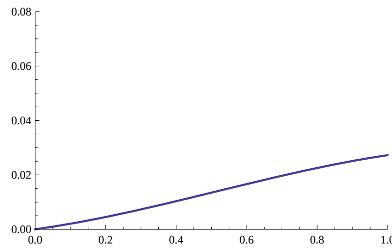
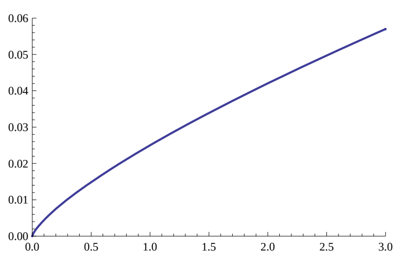


Figure 3 : Neumann series Figure 4 : Modified Neumann series($\varepsilon = 0.017$).

Example 4.3 *Consider the following Fredholm integral equation*

$$\phi(x) = -\frac{x^{0.75}}{40} + x^2 + 0.1\int_0^1 x^{0.75}y\ \phi(y)dy$$

Exact solution is $\phi(x) = x^2$, *and via* $\varepsilon = 0.0035$, *the graphs of the errors for* x_{10} *in both methods are as follows:*



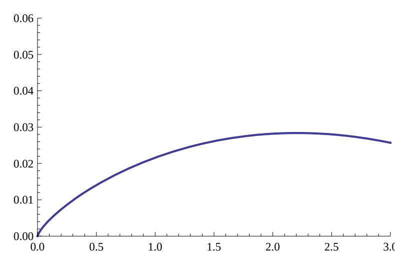


Figure 5 : Error of Neumann series Figure 6 : Modified Neumann series ($\varepsilon = 0.0035$).

5 Conclusion

In this paper we proposed a perturbation method for solving a Fredholm integral equations and we introduced a Modifed Neumann series. In this work we show the decrement of the bound of error.

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