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The Convergence of the Law of the Logarithm Involving Negatively Associated Random Fields[1]

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Abstract

In this paper, the convergence of negatively associated random fields is investigated, two sufficiency conditions for the law of the logarithum are obtained by the Rosenthal and maximal Rosenthal inequalities .

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1. Introduction

The notion of negatively associated (NA) was introduced by Joag-Dev and Proschan in[3] as follows.

Definition 1.1. A field $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ is called negatively associated, if

$$cov(f(X_{\boldsymbol{i}}, \boldsymbol{i} \in \boldsymbol{S}), g(X_{\boldsymbol{j}}, \boldsymbol{j} \in \boldsymbol{T})) \leq 0 \tag{1.1}$$

for each pair of disjoint subsets $\boldsymbol{S}$, $\boldsymbol{T}$ in Z_+^d, where $f(X_{\boldsymbol{i}}, \boldsymbol{i} \in \boldsymbol{S})$ and $g(X_{\boldsymbol{j}}, \boldsymbol{j} \in \boldsymbol{T})$ are any pair of coordinate-wise increasing functions with $Ef^2(X_{\boldsymbol{i}}, \boldsymbol{i} \in \boldsymbol{S}) < \infty$ and $Eg^2(X_{\boldsymbol{j}}, \boldsymbol{j} \in \boldsymbol{T}) < \infty$.

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First we recall some notations . Let d be a positive integer, Z_+^d denote $d-$dimensional lattice points, and points in Z_+^d are denoted by $\boldsymbol{m}, \boldsymbol{n}$ etc. For $\boldsymbol{n} = (n_1, \cdots, n_d) \in Z_+^d$, we define $|\boldsymbol{n}| = \prod_{i=1}^d n_i$, and $\boldsymbol{n} \to \infty$ is interpreted as $n_i \to \infty, i = 1, 2, \cdots, d$. And also for $\boldsymbol{t}, \boldsymbol{s} \in Z_+^d$, $\delta \in Z_+$, let $\boldsymbol{ts} = (t_1 s_1, \ldots, t_d s_d)$, $\boldsymbol{t} + \boldsymbol{s} = (t_1 + s_1, \ldots, t_d + s_d)$, $\boldsymbol{t}\delta = (t_1\delta, \ldots, t_d\delta)$ and $\boldsymbol{t} + \delta = (t_1 + \delta, \ldots, t_d + \delta)$.

Negatively associated is one of the most important concept in probability theory. It can be used in multinomial statistical analysis and reliability theory(see [1,3,4,14]).

In the case of $d = 1$, we refer to Newman [7] for the central limit theorem, Matula [5]for the three series theorem, Roussas [10] for the Hoeffding inequality, Shao [12] for the Rosenth-type inequality and the Kolmogorov exponential inequality, Shao and Su [13] for the law of the iterated logarithm.

In the case of $d \geq 2$, Rossas [11] studies the central limit theorem for weak stationary NA random fields, Zhang[15] investigated the limit theorem for asymptotically negatively dependent random variables, Zhang and Wen [16] obtained the weak convergence for a NA random fields with only finite second moment as follows.

Theorem A.(see [16]) Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in N^d\}$ be a field of stationary centered NA random variables with $0 < EX_{\boldsymbol{e}}^2 < \infty$.

If define

$$W_n(\boldsymbol{t}) = \frac{S[\boldsymbol{n}t]}{\sqrt{|\boldsymbol{n}|}}, \qquad \boldsymbol{t} \in [0,1]^d.$$

Then

$$\frac{ES_{\boldsymbol{n}}^2}{|\boldsymbol{n}|} \to \sigma^2 < \infty, \qquad \boldsymbol{n} \to \infty$$

and

$$W_{\boldsymbol{n}} \Rightarrow \sigma W, \qquad \boldsymbol{n} \to \infty$$

in the space $D_{[0,1]^d}$ endowed with the Skorohod topology, where $\{W(\boldsymbol{t}), \boldsymbol{t} \in R_+^d\}$ is a $d-$dimensional standard winner process. If $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z^d\}$ is the extension of $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in N^d\}$, then

$$\sigma^2 = varX_{\boldsymbol{n}} + \sum_{\boldsymbol{n} \neq \boldsymbol{e}, \boldsymbol{p} \in Z^d} cov(X_{\boldsymbol{e}}, X_{\boldsymbol{p}}).$$

A.Gut conjecture the following result in [2].

Theorem B. Let X and $\{X_n, n \geq 1\}$ are independent identical distribution random variables, if $EX^2(\lg|X|)^{-1+\eta} < \infty$ for some $\eta > 0$ and $EX = 0$, then

$$\sum_{n \geq 1} \frac{1}{n} P(|S_n| \geq \varepsilon (n \lg n)^{\frac{1}{2}}) < \infty \tag{1.2}$$

and

$$\sum_{n\geq 1}\frac{1}{n}P(\max_{1\leq k\leq n}|S_k|\geq \varepsilon(n\lg n)^{\frac{1}{2}})<\infty \tag{1.3}$$

for any $\varepsilon>0$. Conversely, (1.3) $\Rightarrow$ (1.2) $\Rightarrow$ $\frac{EX^2}{\lg|X|}<\infty$ and $EX=0$.

Here $\lg x=\max\{1,\log x\}$.

In this paper, we study the convergence of the law of the logarithm involving negatively associated fields, and obtain the following results.

Theorem1.1. Let $\{X_{\boldsymbol{n}},\boldsymbol{n}\in Z_+^d\}(d\geq 1)$ be a field of negatively associated random variables with identical distribution and $EX_{\boldsymbol{n}}=0$, if for any $\alpha>d$, there exists $0<\eta\leq 1$, such that $E[\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}}]<\infty$, then

$$\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(|S_{\boldsymbol{n}}|\geq \varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha})<\infty \tag{1.4}$$

and

$$\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{k}\leq\boldsymbol{n}}|S_{\boldsymbol{k}}|\geq \varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha})<\infty \tag{1.5}$$

for any $\varepsilon>0$. Conversely, (1.5) $\Rightarrow$ (1.4) $\Rightarrow$ $E[\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{2\alpha}}]<\infty$.

Theorem 1.2. Let $\{X_{\boldsymbol{n}},\boldsymbol{n}\in Z_+^d\}(d\geq 1)$ be a field of negatively associated random variables with identical distribution and $EX_{\boldsymbol{n}}=0$, if for any $\alpha>d$, there exists $0<\eta\leq 1$, such that $E[\frac{X_{\boldsymbol{1}}^2}{(\lg\lg|X_{\boldsymbol{1}}|)^{1-\eta}}]<\infty$, then

$$\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|\lg|\boldsymbol{n}|}P(\max_{\boldsymbol{k}\leq\boldsymbol{n}}|S_{\boldsymbol{k}}|\geq \varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg\lg|\boldsymbol{n}|)^{\alpha})<\infty \tag{1.6}$$

for any $\varepsilon>0$.

2. Lemmas and Proof of Theorems

In order to prove the main results of this article, we require to establish and introduce some lemmas in this section.

Lemma 2.1.(see[10]) Let $\{X_{\boldsymbol{n}},\boldsymbol{n}\in Z_+^d\}(d\geq 1)$ be a field of negatively associated random variables with identical distribution and $EX_{\boldsymbol{n}}=0$, then for all $p\geq 2$, there exists a positive constant C such that

$$E\left|\sum_{\boldsymbol{k}\in A}X_{\boldsymbol{k}}\right|^p\leq C\left[\left(\sum_{\boldsymbol{k}\in A}EX_{\boldsymbol{k}}^2\right)^{p/2}+\sum_{\boldsymbol{k}\in A}E|X_{\boldsymbol{k}}|^p\right] \tag{2.1}$$

and

$$E\max_{\boldsymbol{k}\leq\boldsymbol{n}}|S_{\boldsymbol{k}}|^p\leq C\left[\left(E\max_{\boldsymbol{k}\leq\boldsymbol{n}}|S_{\boldsymbol{k}}|\right)^p+\left(\sum_{\boldsymbol{k}\leq\boldsymbol{n}}EX_{\boldsymbol{k}}^2\right)^{p/2}+\sum_{\boldsymbol{k}\leq\boldsymbol{n}}E|X_{\boldsymbol{k}}|^p\right] \tag{2.2}$$

for any $A \in Z_+^d$ and $\boldsymbol{n} \in Z_+^d$.

Lemma 2.2. Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}(d \geq 1)$ be a field of negatively associated random variables with identical distribution and $EX_{\boldsymbol{n}} = 0$, then for all $q \geq 2$, there exists a positive constant C such that

$$E \max_{\boldsymbol{k} \leq \boldsymbol{n}} |S_{\boldsymbol{k}}|^q \leq C \left[\sum_{\boldsymbol{k} \leq \boldsymbol{n}} E|X_{\boldsymbol{k}}|^q + (\lg |\boldsymbol{n}|)^{qd} \left(\sum_{\boldsymbol{k} \leq \boldsymbol{n}} EX_{\boldsymbol{k}}^2 \right)^{q/2} \right] \tag{2.3}$$

Proof. We now estimate $E \max_{\boldsymbol{k} \leq \boldsymbol{n}} |S_{\boldsymbol{k}}|$ in (2.2). Taking $p = 2$ in (2.1) we get

$$E \left(\sum_{\boldsymbol{j} \leq \boldsymbol{k} \leq \boldsymbol{m}} |X_{\boldsymbol{k}}| \right)^2 \leq C \sum_{\boldsymbol{j} \leq \boldsymbol{k} \leq \boldsymbol{m}} X_{\boldsymbol{k}}^2 \tag{2.4}$$

for $\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{m} \leq \boldsymbol{n}$.

(2.2) and (2.4) together with Móreicz[12, Theorem 8] lead to

$$E \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} S_{\boldsymbol{k}}^2 \leq C \prod_{k=1}^{d} (\lg n_k)^2 \sum_{\boldsymbol{k} \leq \boldsymbol{n}} EX_{\boldsymbol{k}}^2 \leq C(\lg |\boldsymbol{n}|)^{2d} \sum_{\boldsymbol{k} \leq \boldsymbol{n}} EX_{\boldsymbol{k}}^2. \tag{2.5}$$

(2.5) implies that

$$(E \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} |S_{\boldsymbol{k}}|)^q \leq (E \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} S_{\boldsymbol{k}}^2)^{q/2} \leq C(\lg |\boldsymbol{n}|)^{qd} \left(\sum_{\boldsymbol{k} \leq \boldsymbol{n}} EX_{\boldsymbol{k}}^2 \right)^{q/2}. \tag{2.6}$$

Now Lemma 2.2 follows from (2.2) and (2.6).

Lemma 2.3. Let $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}(d \geq 1)$ be a field of negatively associated random variables with identical distribution, if $P(\max_{\boldsymbol{i} \leq \boldsymbol{n}} |X_{\boldsymbol{i}}| > a_{\boldsymbol{n}}) \to 0$ as $\boldsymbol{n} \to \infty$ for $a_{\boldsymbol{n}} > 0$, then there exists a positive constant C such that

$$|\boldsymbol{n}| P(|X_{\boldsymbol{1}}| > a_{\boldsymbol{n}}) \leq C P(\max_{\boldsymbol{i} \leq \boldsymbol{n}} |X_{\boldsymbol{i}}| > a_{\boldsymbol{n}}) \tag{2.7}$$

for sufficient large $\boldsymbol{n}$.

Proof. It is well-known that

$$P(\max_{\boldsymbol{i} \leq \boldsymbol{n}} |X_{\boldsymbol{i}}| > a_{\boldsymbol{n}}) = \sum_{\boldsymbol{j} \leq \boldsymbol{n}} P(|X_{\boldsymbol{j}}| > a_{\boldsymbol{n}}, \max_{\boldsymbol{i} \leq \boldsymbol{j}-\boldsymbol{1}} |X_{\boldsymbol{i}}| \leq a_{\boldsymbol{n}}). \tag{2.8}$$

From (2.8) and equidistribution of $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in Z_+^d\}$ we have

$$|\boldsymbol{n}| P(|X_{\boldsymbol{1}}| > a_{\boldsymbol{n}}) = P(\max_{\boldsymbol{i} \leq \boldsymbol{n}} |X_{\boldsymbol{i}}| > a_{\boldsymbol{n}}) + \sum_{\boldsymbol{j} \leq \boldsymbol{n}} P(|X_{\boldsymbol{j}}| > a_{\boldsymbol{n}}, \max_{\boldsymbol{i} \leq \boldsymbol{j}-\boldsymbol{1}} |X_{\boldsymbol{i}}| > a_{\boldsymbol{n}}). \tag{2.9}$$

By centering $\sum_{\boldsymbol{j}\le\boldsymbol{n}} P(|X_{\boldsymbol{j}}|>a_{\boldsymbol{n}}, \max_{\boldsymbol{i}\le\boldsymbol{j}-\boldsymbol{1}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})$, we get

$$\begin{aligned}
&\sum_{\boldsymbol{j}\le\boldsymbol{n}} P(|X_{\boldsymbol{j}}|>a_{\boldsymbol{n}}, \max_{\boldsymbol{i}\le\boldsymbol{j}-\boldsymbol{1}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})\\
&\le E\sum_{\boldsymbol{j}\le\boldsymbol{n}}[I(|X_{\boldsymbol{j}}|>a_{\boldsymbol{n}})-P(|X_{\boldsymbol{1}}|>a_{\boldsymbol{n}})]I(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})\\
&\qquad +|\boldsymbol{n}|P(|X_{\boldsymbol{1}}|>a_{\boldsymbol{n}})P(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})\\
&= I_1+I_2.
\end{aligned}\tag{2.10}$$

Now we estimate I_1, by Cauchy-Schuwarz Theorem and taking $p=2$ in (2.1) we get

$$\begin{aligned}
|I_1| &\le \left(var[\sum_{\boldsymbol{j}\le\boldsymbol{n}} I(|X_{\boldsymbol{j}}|>a_{\boldsymbol{n}})]\cdot P(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})\right)^{1/2}\\
&\le (C|\boldsymbol{n}|P(|X_{\boldsymbol{1}}|>a_{\boldsymbol{n}})\cdot P(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}}))^{\frac{1}{2}}\\
&\le \frac{1}{4}|\boldsymbol{n}|P(|X_{\boldsymbol{1}}|>a_{\boldsymbol{n}})+CP(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}}).
\end{aligned}\tag{2.11}$$

The last inequality follows from the elementary inequality: $(ab)^{\frac{1}{2}}\le\frac{1}{4}a+b$ for $a>0, b>0$.

Then (2.9)-(2.11) lead to

$$\frac{3}{4}|\boldsymbol{n}|P(|X_{\boldsymbol{1}}|>a_{\boldsymbol{n}})\le 2CP(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})+|\boldsymbol{n}|P(|X_{\boldsymbol{1}}|>a_{\boldsymbol{n}})P(\max_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}}).\tag{2.12}$$

Therefore (2.7) follows from the assumption $P(\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>a_{\boldsymbol{n}})\to 0$ and (2.12).

Lemma 2.4. Under the same assumption of Theorem 1.1. If $0<\beta<2\alpha+\eta-1$, then

$$\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})<\infty.\tag{2.13}$$

Proof. By the assumption of equidistribution and $E[\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}}]<\infty$ together with simply computation, we have

$$\begin{aligned}
&\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\\
&\le\sum_{\boldsymbol{n}}P(|X_{\boldsymbol{1}}|>|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\\
&\le\sum_{\mathbf{n}}P\left(\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}}>\frac{|\boldsymbol{n}|(\lg|\boldsymbol{n}|)^{2(\alpha-\beta)}}{(\lg(|\boldsymbol{n}|^{\frac{1}{2}})(\lg|\boldsymbol{n}|)^{\alpha-\beta})^{1-\eta}}\right)
\end{aligned}$$

$$\leq \sum_{\boldsymbol{n}} P\left(\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}} \geq C|\boldsymbol{n}|\right)$$
$$\leq E\left(\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}}\right) < \infty.$$

Hence Lemma2.4 is true.

Lemma 2.5. Under the same assumption of Theorem 1.1. Let $X_{\boldsymbol{k},\boldsymbol{n}} = X_{\boldsymbol{k}} I(|X_{\boldsymbol{k}}| < |\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})$ and $S_{\boldsymbol{m},\boldsymbol{n}} = \sum_{\boldsymbol{k}\leq\boldsymbol{m}} X_{\boldsymbol{k},\boldsymbol{n}}$. If $0 < \beta < 2\alpha + \eta - 1$, then

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} P(\max_{\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{n}} |S_{\boldsymbol{k},\boldsymbol{n}}| > \varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha}) < \infty \tag{2.14}$$

for any $\varepsilon > 0$.

Proof. We first proof that

$$|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{-\alpha} \max_{\boldsymbol{k}\leq\boldsymbol{n}} |E\sum_{\boldsymbol{i}\leq\boldsymbol{k}} X_{\boldsymbol{i},\boldsymbol{n}}| \to 0 \tag{2.15}$$

as $\boldsymbol{n} \to \infty$.

$EX_{\boldsymbol{n}} = 0$ implies that

$$X_{\boldsymbol{k},\boldsymbol{n}} = X_{\boldsymbol{k}} I(|X_{\boldsymbol{n}}| < |\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta}) = X_{\boldsymbol{k}} I(|X_{\boldsymbol{k}}| > |\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta}), \tag{2.16}$$

this leads to

$$\begin{aligned}
&|\boldsymbol{n}|^{-\frac{1}{2}}(\lg|\boldsymbol{n}|)^{-\alpha} \max_{\boldsymbol{k}\leq\boldsymbol{n}} |E\sum_{\boldsymbol{i}\leq\boldsymbol{k}} X_{\boldsymbol{i},\boldsymbol{n}}| \\
&= |\boldsymbol{n}|^{-\frac{1}{2}}(\lg|\boldsymbol{n}|)^{-\alpha}|\boldsymbol{n}|E\left[|X_{\boldsymbol{1}}|I(|X_{\boldsymbol{1}}| < |\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\right] \\
&\leq |\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{-\alpha}E\left[\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}} \cdot \frac{|X_{\boldsymbol{1}}|^{-1}}{(\lg|\boldsymbol{n}|)^{\eta-1}} I(|X_{\boldsymbol{1}}| < |\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\right] \\
&\leq C(\lg|\boldsymbol{n}|)^{\beta-2\alpha+1-\eta}.
\end{aligned} \tag{2.17}$$

Now (2.15) follows from (2.17) and the assumption $0 < \beta < 2\alpha + \eta - 1$.

In order to prove (2.14), by (2.15) we only need to prove that

$$\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} P\left(\max_{\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{n}} |\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{k}} X_{\boldsymbol{i},\boldsymbol{n}} - \sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{k}} EX_{\boldsymbol{i},\boldsymbol{n}}| \geq \varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha}\right) < \infty. \tag{2.18}$$

Taking $q \geq 2$ in Lemma 2.2, we get

$$\begin{aligned}
&\sum_{\boldsymbol{n}} \frac{1}{|\boldsymbol{n}|} P(\max_{\boldsymbol{1}\leq\boldsymbol{k}\leq\boldsymbol{n}} |\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{k}} X_{\boldsymbol{i},\boldsymbol{n}} - \sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{k}} EX_{\boldsymbol{i},\boldsymbol{n}}| \geq \varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha}) \\
&\leq \sum_{\boldsymbol{n}} \varepsilon C \frac{1}{|\boldsymbol{n}|} \cdot |\boldsymbol{n}|^{-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q}[\sum_{\boldsymbol{i}\leq\boldsymbol{n}} E|X_{\boldsymbol{i},\boldsymbol{n}}|^q + (\lg|\boldsymbol{n}|)^{qd}(\sum_{\boldsymbol{i}\leq\boldsymbol{n}} E|X_{\boldsymbol{i},\boldsymbol{n}}|^2)^{\frac{q}{2}}]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{n}} C|\boldsymbol{n}|^{-1-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q}\sum_{\boldsymbol{i}\le\boldsymbol{n}} E|X_{\boldsymbol{i},\boldsymbol{n}}|^q \\
&\qquad + \sum_{\boldsymbol{n}} C|\boldsymbol{n}|^{-1-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q+qd}\Big(\sum_{\boldsymbol{i}\le\boldsymbol{n}} E|X_{\boldsymbol{i},\boldsymbol{n}}|^2\Big)^{\frac{q}{2}} \\
&= I_3 + I_4. \qquad (2.19)
\end{aligned}
$$

Next we estimate I_3 and I_4,

$$
\begin{aligned}
I_3 &\le \sum_{\boldsymbol{n}} C|\boldsymbol{n}|^{-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q}E[|X_{\boldsymbol{1}}|^q I(|X_{\boldsymbol{1}}|<|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha-\beta})] \\
&\le \sum_{\boldsymbol{n}} C|\boldsymbol{n}|^{-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q}E\left[\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}}\cdot\frac{X_{\boldsymbol{1}}^{q-2}}{(\lg|\boldsymbol{n}|)^{\eta-1}}I(|X_{\boldsymbol{1}}|<|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\right] \\
&\le \sum_{\boldsymbol{n}} C|\boldsymbol{n}|^{-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q}|\boldsymbol{n}|^{\frac{q-2}{2}}(\lg|\boldsymbol{n}|)^{1-\eta+(q-2)(\alpha-\beta)} \\
&\le \sum_{\boldsymbol{n}} C\frac{1}{|\boldsymbol{n}|}(\lg|\boldsymbol{n}|)^{-\alpha q+1-\eta+(q-2)(\alpha-\beta)}. \qquad (2.20)
\end{aligned}
$$

Now (2.20) together with the assumption $0<\beta<2\alpha+\eta-1$ and $q\ge 2$ lead to $I_3<\infty$.

$$
\begin{aligned}
I_4 &\le \sum_{\boldsymbol{n}} C|\boldsymbol{n}|^{-1-\frac{q}{2}}(\lg|\boldsymbol{n}|)^{-\alpha q+qd}|\boldsymbol{n}|^{\frac{q}{2}}(E|X_{\boldsymbol{i},\boldsymbol{n}}|^2)^{\frac{q}{2}} \\
&= \sum_{\boldsymbol{n}} C\frac{1}{|\boldsymbol{n}|}(\lg|\boldsymbol{n}|)^{-\alpha q+qd}\left(EX_{\boldsymbol{1}}^2 I(|X_{\boldsymbol{1}}|<|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\right)^{\frac{q}{2}} \\
&= \sum_{\boldsymbol{n}} C\frac{1}{|\boldsymbol{n}|}(\lg|\boldsymbol{n}|)^{-\alpha q+qd}\left(E\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{1-\eta}}\cdot(\lg|X_{\boldsymbol{1}}|)^{1-\eta}I(|X_{\boldsymbol{1}}|<|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha-\beta})\right)^{\frac{q}{2}} \\
&\le \sum_{\boldsymbol{n}} C\frac{1}{|\boldsymbol{n}|}(\lg|\boldsymbol{n}|)^{-\alpha q+qd}(\lg|\boldsymbol{n}|)^{\frac{q}{2}(1-\eta)} \\
&= \sum_{\boldsymbol{n}} C\frac{1}{|\boldsymbol{n}|}(\lg|\boldsymbol{n}|)^{-\alpha q+qd+\frac{q}{2}(1-\eta)}. \qquad (2.21)
\end{aligned}
$$

The assumption $\alpha>d$ implies that there exists $0<\eta\le 1$, such that $\frac{1-\eta}{2}<\alpha-d$, this and (2.21) lead to $I_4<\infty$.

Hence (2.14) follows from (2.19)-(2.21).

Proof of Theorem1.1. It is obvious that (1.5) $\Rightarrow$ (1.4), so we only need to prove (1.5). Recall that $X_{\boldsymbol{k},\boldsymbol{n}} = X_{\boldsymbol{k}}I(|X_{\boldsymbol{k}}|<|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha-\beta})$, $S_{\boldsymbol{m},\boldsymbol{n}} = \sum_{\boldsymbol{k}\le\boldsymbol{m}} X_{\boldsymbol{k},\boldsymbol{n}}$ and $0<\beta<2\alpha+\eta-1$, it is easy to see that

$$
\begin{aligned}
&\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{n}}|S_{\mathbf{k}}|\ge\varepsilon|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha}) \\
&= \sum_{\boldsymbol{n}}\frac{1}{\boldsymbol{n}|}P(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{n}}|S_{\boldsymbol{k}}|\ge\varepsilon|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha},\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>|\boldsymbol{n}|^{\frac12}(\lg|\boldsymbol{n}|)^{\alpha-\beta})
\end{aligned}
$$

$$+\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{n}}|S_{\boldsymbol{n}}|\ge\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha},\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|\le|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})$$
$$\le\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha-\beta})$$
$$+\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{n}}|S_{\boldsymbol{k},\boldsymbol{n}}|\ge\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha}). \tag{2.22}$$

Now (1.5) follows from Lemma 2.4, Lemma 2.5 and (2.22).

Conversely, $X_{\boldsymbol{k}}\le|S_{\boldsymbol{k}}|-|S_{\boldsymbol{k}-\boldsymbol{1}}|$ and (1.5) yield that

$$\sum_{\boldsymbol{n}}\frac{1}{|\boldsymbol{n}|}P(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{n}}|X_{\boldsymbol{k}}|\ge\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha})<\infty, \tag{2.23}$$

this implies

$$P(\max_{\boldsymbol{1}\le\boldsymbol{k}\le\boldsymbol{n}}|X_{\boldsymbol{k}}|\ge\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha})\to 0. \tag{2.24}$$

By Lemma 2.3 and (2.24) we get

$$|\boldsymbol{n}|P(|X_{\boldsymbol{1}}|>\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)^{\alpha})\le CP(\max_{\boldsymbol{i}\le\boldsymbol{n}}|X_{\boldsymbol{i}}|>\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|)) \tag{2.25}$$

for large $\boldsymbol{n}$.

Now (2.23) and (2.25) lead to

$$\sum_{\boldsymbol{n}}P(|X_{\boldsymbol{1}}|\ge\varepsilon|\boldsymbol{n}|^{\frac{1}{2}}(\lg|\boldsymbol{n}|))<\infty, \tag{2.26}$$

that is

$$E\left[\frac{X_{\boldsymbol{1}}^2}{(\lg|X_{\boldsymbol{1}}|)^{2\alpha}}\right]<\infty.$$

This complete the proof the Theorem1.1. Making use of the parallel method as in proof of Theorem 1.1, we can prove Theorem 1.2.

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