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A Goodness of Fit Approach to Monotone Variance Residual Life Class of Life Distributions

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Abstract

Based on the goodness of fit approach, a new test is presented for testing exponentiality versus decreasing (increasing) variance remaining life distribution DVRL (IVRL). The percentiles of these test are tabulated for sample sizes n=5(1)6(2)50. It is shown that the proposed test is simple, has high relative efficiency for some commonly used alternatives and enjoys a good power. An example in medical science is considered as a practical application of the proposed test.

Keywords: DVRL (IVRL) class of life distribution, exponentiality, efficiency, asymptotic normality

1 Introduction

The variance residual life (VRL) distributions are useful in many areas including biometry, actuarial science and reliability. Let T denote the life time of an equipment distribution function $F(x)$, survival function $\overline{F} = 1-F$, mean life $\mu = \int_0^\infty \overline{F}(u)du$ and variance life $\sigma^2 = var(T)$ both assumed finite. The mean residual life (MRL) and the variance residual life (VRL) are defined as the following:

$$\mu(t) = E\{T - t|T \geq t\} = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)}, \qquad t \geq 0, \tag{1.1}$$

and

$$\sigma^2(t) = var\{T - t|T \geq t\} = var\{T|T \geq t\}. \tag{1.2}$$

Consider $E[U^2/t] = -\int_0^\infty u^2 d\overline{F}(u/t)$, integrating by parts, one have

$$\sigma^2(t) = \frac{2}{\overline{F}(t)} \int_t^\infty \int_y^\infty \overline{F}(x)dxdy - \mu^2(t), \tag{1.3}$$

let $\nu(y) = \int_y^\infty \overline{F}(x)dx$ and $\Gamma(t) = \int_t^\infty \nu(y)dy$, then (1.1) and (1.3) become as the following:

$$\mu(t) = \frac{\nu(t)}{\overline{F}(t)}, \tag{1.4}$$

and

$$\sigma^2(t) = \frac{2}{\overline{F}(t)} \int_t^\infty \nu(y)dy = \frac{2\Gamma(t)}{\overline{F}(t)} - \mu^2(t). \tag{1.5}$$

or

$$\sigma^2(t) = \frac{2\overline{F}(t)\Gamma(t) - \mu^2(t)}{\overline{F}^2(t)}. \tag{1.6}$$

A distribution function F is said to be a decreasing (increasing) variance residual life DVRL (IVRL) if $\sigma^2(t)$ is nondecreasing (nondecreasing) function of t (i.e.$\frac{d\sigma^2(t)}{dt} \leq (\geq)0$). Differentiating (1.4) and (1.5) with respect to t, we have

$$\frac{d\mu(t)}{dt} = -1 + \nu(t)\mu(t). \tag{1.7}$$

$$\frac{d\sigma^2(t)}{dt} = \frac{2f(t)\Gamma(t)}{\overline{F}^2(t)} - \frac{2\nu(t)}{\overline{F}(t)} - 2\mu(t)\frac{d\mu(t)}{dt}. \tag{1.8}$$

Using (1.4) and (1.7) in (1.8), we obtain

$$\frac{d\sigma^2(t)}{dt} = r(t)[\sigma^2(t) - \mu^2(t)],$$

where $r(t) = \frac{f(t)}{\overline{F}(t)}$. Since $r(t)$ is nonnegative for all t, let us recall that $\overline{F}(t)$ is DVRL (IVRL) if $\sigma^2(t) \leq (\geq)\mu^2(t)$, by using (1.16) this implies that $\overline{F}(t)$ is DVRL (IVRL) if

$$2\overline{F}(t)\Gamma(t) \leq (\geq)\ 2\nu^2(t).$$

Now, we have the following definition

Definition (1.1): A life distribution F, with $F(0) = 0$ and its survival function $\overline{F}$ is said to have DVRL (IVRL) class of life distributions if

$$\overline{F}(t)\Gamma(t) \leq (\geq)\nu^2(t). \tag{1.9}$$

Launcer (1987), Gupta (1987), Gupta et al (1987), Kanjo (1996) and Gupta and Kirmani (2000) are studied characterization of this class and used it to

find better bounds on moments and survival function. The null distribution for DVRL(IVRL) is the exponential. Thus we often encounter testing H_0 : A life distribution is exponential versus H_1 : It is DVRL (IVRL) and not exponential. This testing problem was investigated by Kanwar and Madhu (1991), Fango(1996) and recently by Abu-Youssef (2004, 2007). However in contrast to goodness of fit problems, where the test statistics is based on a measure of departure from H_0 that depends on both H_0 and H_1. Most tests of life testing setting included those refereed above did not use the null distribution in devising the test statistics, which resulted in test statistics that are often difficult to work with require programming to evaluate.
Recently Ahmad et al(2001), El-Bassiouny and El-Wasel (2003) and Abu-Youssef(2007) were used a new methodology for testing by incorporating both H_0 and H_1 in devising the test statistics for testing H_0 against the alternative the life distribution is IFR, NBUC, HNBUE and DMRL classes of life distributions. They obtained very simple statistics that are not asymptotically equivalent in distribution and efficiency to classical procedure but also better in finite sample behaviors. Our goal in this paper is to use similar methodology to obtain a very simple statistics for testing H_0 against H_1. The thread that connects most work mentioned here is that a measure of departure from H_0, which is strictly positive under H_1 and is zero under H_0. Then, a sample version of this measure is used as test statistics and its properties are studied. In section 2, we propose a test statistic, based on the goodness of fit approach, for testing H_0: F is exponential against H_1: F is DVRL (IVRL) and not exponential. We then present Monte Carlo null distribution critical points for sample sizes $n = 5(1)6(2)50$. In section 3 we calculate the efficiency of the test statistic for some common alternatives and compared them to other procedures. In section 4 we give simulated values of the power estimates of the test. Finally an application in medical science was introduced in section 5.

2 Testing DVRL (IVRL) class of life distribution

The test presented have depends on a sample $X_1, X_2, \ldots, X_n$ from a population with distribution F. We wish to test the null hypothesis $H_0 : \overline{F}$ is exponential with mean μ against $H_1 : \overline{F}$ is DVRL (IVRL) class of life distribution and not exponential, using the inequality(1.9), one used the following as a measure of departure from H_0 in favor of H_1:

$$\delta_v = \int_0^\infty [\nu^2(t) - \overline{F}(t)\Gamma(t)]dt. \tag{2.1}$$

But

$$\int_0^\infty \nu^2(y)dy = \int_0^\infty x^2\nu(x)dF(x) + \frac{2}{3}x^2\overline{F}(x)dF(x). \tag{2.2}$$

and

$$\begin{aligned}\int_0^\infty \overline{F}(x)\Gamma(x)dx &= \int_0^\infty x\Gamma(x)dF(x) + \frac{1}{2}\int_0^\infty x^2\nu(x)dF(x) \\ &+ \frac{1}{3}x^3\overline{F}(x)dF(x).\end{aligned} \tag{2.3}$$

Then, from (2.2) and (2.3), the measure in (2.1) becomes as the following:

$$\delta_v = \int_0^\infty [\frac{x^2}{2}\nu(t) + \frac{x^3}{3}\overline{F}(t) - x\Gamma(t)]dF(t). \tag{2.4}$$

Note that under $H_0 : \delta_v = 0$, while under $H_1 : \delta_v > (<)0$
Denote $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding ordered sample and if $F_n = \frac{i}{n}, x\epsilon(X_i, X_{(i+1)]})$ is the empirical distribution function, then $\hat{\overline{F}}_n = \frac{1}{n}\sum_{j=1}^n I(X_j > x)$ is empirical survival function, where $i = 1, 2, ..., n.$ and the empirical functions of δ_v and $\Gamma_n(x)$ are $\hat{\nu_n}(x) = \frac{1}{n}\sum_{j=1}^n (X_j - x)I(X_j > x)$ and $\hat{\Gamma_n}(x) = \frac{1}{2n}\sum_{j=1}^n (X_j - x)^2 I(X_j > x)$ respectively, whereas

$$I(X_j > x) = \begin{cases} 1 & X_j > x \\ 0 & otherwise \end{cases}$$

In a similar fashion, if F_0 denote the exponential distribution, we can take in place of (2.1) or (2.4) the following measure of departure from H_0

$$\delta_{v1} = \int_0^\infty [\frac{x^2}{2}\nu(t) + \frac{x^3}{3}\overline{F}(t) - x\Gamma(t)]dF_0(t), \tag{2.5}$$

for testing the hypothesis that H_0: F is exponential versus H_1: F is DVRL (IVRL) class of life distribution and not exponential. With out loss of generality we take $\mu = 1$ and thus $F_0(x) = 1 - e^{-x}$. In order to derive an expression for $\hat{\delta_{v1_n}}$, we need the following theorem.

Theorem 2.1. Let T be a variable with distribution function F. Then

$$\delta_{v1} = -4 + E[3X - \frac{1}{2}X^2 + e^{-X}(4 + X - \frac{1}{2}X^2 - \frac{1}{3}X^3)]. \tag{2.6}$$

Proof. Note that δ_{v1} in (2.5) be written as the following:

$$\begin{aligned}\delta_{v1} &= \int_0^\infty \frac{x^2}{2}\nu(x)e^{-x}dx + \int_0^\infty \frac{x^3}{3}\overline{F}(x)e^{-x}dx - \int_0^\infty x\Gamma(x)e^{-x}dx \\ &= I1 + I2 - I3 \end{aligned} \tag{2.7}$$

where $I1 = \int_0^\infty \frac{x^2}{2}\nu(x)e^{-x}dx$, $I2 = \int_0^\infty \frac{x^3}{3}\overline{F}(x)dx$ and $I3 = \int_0^\infty x\Gamma(x)dx$
But

$$\begin{aligned} I1 &= \int_0^\infty E(X-x)I(X>x)\frac{x^2}{2}e^{-x}dx \\ &= E\int_0^X (X-x)\frac{x^2}{2}e^{-x}dx \\ &= -3 + E[X + (3 + 2X + \frac{1}{2}X^2)e^{-X}], \end{aligned} \tag{2.8}$$

$$\begin{aligned} I2 &= \int_0^\infty \frac{x^3}{3}EI(X>x)e^{-x}dx \\ &= E\int_0^X \frac{x^3}{3}e^{-x}dx \\ &= 2 + E[(-2 - 2X - X^2 - \frac{1}{3}X^3)e^{-X})], \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} I3 &= \frac{1}{2}\int_0^\infty xE(X-x)^2 I(X>x)e^{-x}dx \\ &= \frac{1}{2}E\int_0^X x(X-x)^2e^{-x}dx \\ &= 3 + E[-2X + \frac{1}{2}X^2 - (3+X)e^{-X}]. \end{aligned} \tag{2.10}$$

Using (2.8), (2.9) and (2.10) in (2.7), we get the result.

Note that: $\delta_{v1} = 0$ under H_0, while it is positive under H_1. Thus based on a random sample $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ from a distribution F. We wish to test H_0 against H_1, we may be testing on its estimate. A direct empirical estimate of δ_{v1} is

$$\hat{\delta}_{v1_n} = -4 + \frac{1}{n}\sum_{i=1}^n \left\{3X_i - \frac{1}{2}X_i^2 + e^{-X_i}(4 + X_i - \frac{1}{2}X_i^2 - \frac{1}{3}X_i^3)\right\}. \tag{2.11}$$

To make the test scale invariant, we take

$$\hat{\Delta}_{v1_n} = \frac{\hat{\delta}_{v1_n}}{\overline{X}^3} \tag{2.12}$$

Theorem 3.1. As $n \to \infty$, $\sqrt{n}(\hat{\Delta}_{v1_n} - \Delta_{v1_n})$ is asymptotically normal with mean 0 and variance σ^2 where σ^2 is given in (2.13). Under H_0:$\sigma_0^2 = 0.27869$.

Proof. Since $\hat{\Delta}_{V1_n}$ and $\frac{\hat{\Delta}_{V1n}}{\mu^3}$ have the same limiting distribution, we use $\sqrt{n}(\hat{\delta}_{v1_n} - \delta_{v1_n})$. Noting that $\hat{\delta}_{V1_n}$ is just an average, it is straightforward by using the central limit theorem the result follows. For the variance

$$\sigma^2 = E[-4 + 3X - \frac{1}{2}X^2 + e^{-4X}(4 + X - \frac{1}{2}X^2 - \frac{1}{2}X^3)]^2. \qquad (2.13)$$

Under $H_0, \Delta_{v1} = 0$ and

$$\sigma_0^2 = \int_0^\infty [-4 + 3X - \frac{1}{2}X^2 + e^{-X}(4 + X - \frac{1}{2}X^2 - \frac{1}{2}X^3)]^2]e^{-x}dx = 0.27869.$$

Then the theorem is proved.

3 Monte carlo null distribution critical points for $\hat{\Delta}_{F_n}$ test

In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. we have simulated the upper percentile points for 90%, 95%, 99%. Table (3.1) gives these percentile points of statistic $\hat{\Delta}_{v1_n}$ in (2.12) and the calculations are based on 5000 simulated samples of sizes $n = 5(1)6(2)50$. The percentile values change slowly as n increase.

Table 3.1 Critical Values of $\hat{\Delta}_{v1_n}$

n	90%	95%	99%
5	.0209	.0704	.3085
6	.0209	.0715	.2614
8	.0219	.0643	.2191
10	.0206	.0546	.1553
12	.0215	.0553	.1398
14	.0207	.0517	.1358
16	.0203	.0504	.1196
18	.0222	.0529	.1135
20	.0222	.0503	.1113
22	.0200	.0438	.0930
24	.0200	.0423	.0890
26	.0194	.0427	.0950
28	.0202	.0410	.0841
30	.0196	.0408	.0824
32	.0204	.0424	.0847
34	.0190	.0380	.0796
36	.0194	.0372	.0853
38	.0187	.0375	.0775
40	.0208	.0382	.0731
42	.0191	.0389	.0776
44	.0208	.0391	.0728
46	.0183	.0351	.0716
48	.0182	.0375	.0666
50	.0176	.0361	.0649

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{v1_n}/\sigma_0^2$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$.

4 Asymptotic relative efficiency (Are)

We compare our test $\hat{\Delta}_{v1_n}$to tests $\hat{\Delta}_{v_n}$ and $\hat{\Delta}_{kv_n}$ presented by Abu-Youssef (2004, 2007) for DVR classes of life distributions. The comparisons are achieved by using Pitman asymptotic relative efficiency (PARE), which is defined as follows:

Let T_{1_n} and T_{2_n} be two statistics for testing H_o: $F_\theta \epsilon \{F_{\theta_x}\}, \theta_n = \theta + \frac{c}{\sqrt{n}}$ with c an arbitrary constant, then PARE of T_{1_n} relative to T_{2_n} is defined by

$$e(T_{1_n}, T_{2_n}) = \frac{\mu_1^{'}(\theta_o)}{\sigma_1(\theta_o)} / \frac{\mu_2^{'}(\theta_o)}{\sigma_2(\theta_o)}$$

where $\mu_i^{\cdot}(\theta_o) = \lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{in})_{\to\theta_o}$ and $\sigma_i^2(\theta_o) = \lim_{n\to\infty} VarE(T_{in}), i = 1, 2.$ Two of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i) Linear failure rate family : $\bar{F}_{1\theta} = e^{-x-\frac{\theta x^2}{2}}, \quad x > 0, \theta > 0$
(ii) Weibull family : $\bar{F}_{2\theta} = e^{-x^\theta}, \quad x \geq 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and $\theta = 1$ for Weibull family. Direct calculations of PAE of $\hat{\Delta}_{v1_n}$, $\hat{\Delta}_{v_n}$ and $\hat{\Delta}_{kv_n}$ are summarized in Table (4.1).

Table 4.1 PAE of $\hat{\Delta}_{v1_n}$, $\hat{\Delta}_{v_n}$ and $\hat{\Delta}_{kv_n}$

Distribution	$\hat{\Delta}_{v1_n}$	$\hat{\Delta}_{V_n}$	$\hat{\Delta}_{kv_n}$
F_1 (Linear failure rate)	1.54	0.91	0.94
F_2 (Weibull)	0.74	1.83	1,89

The efficiencies in Table (4.1) show clearly our statistic ($\hat{\Delta}_{v1_n}$) performs better than $\hat{\Delta}_{v_n}$ for F_1, also it performs better than $\hat{\Delta}_{kv_n}$ for F_1 .

In table 4.2 we give PARE's of $\hat{\Delta}_{v1_n}$ with respect to $\hat{\Delta}_{v_n}$ and $\hat{\Delta}_{kv_n}$ whose PAE are mentioned in table 4.1.

Table 4.2 PARE of $\hat{\Delta}_{v1_n}$ with respect to $\hat{\Delta}_{v_n}$ and $\hat{\Delta}_{kv_n}$

Distribution	$e_{F_i}(\hat{\Delta}_{v1_n},\hat{\Delta}_{v_n})$	$e_{F_i}(\hat{\Delta}_{v1_n},\hat{\Delta}_{kv_n})$
F_1 (Linear failure rate)	1.69	1.64
F_2(Weibull)	0.40	0.39

It is clear from Table 4.2 that the statistic $\hat{\Delta}_{v1_n}$ performs well for $\overline{F}_1$ and it is more efficient. Finally, the power of the test statistics $\hat{\Delta}_{v1_n}$ is considered for 95% percentiles in Table 4.3 for two of the most commonly used alternatives [see Hollander and Proschan (1975)], they are:

(i) Linear failure rate : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}, \quad x > 0, \theta > 0$
ii) Weibull family : $\bar{F}_{3\theta} = e^{-x^\theta}, \quad x \geq 0, \theta > 0$

These distributions are reduced to exponential distribution for appropriate values of θ .

Table 4.3 Power Estimate of $\hat{\Delta}_{v1n}$

Distribution	θ	Sample Size			
		n=10	n=20	n=30	
F_1	1	0.980	0.993	1.000	
Linear failure	2	1.000	1.000	1.000	
rate	3	1.000	1.000	1.000	
F_2	1	0.737	0.785	0.785	
Weibull	2	0.845	0.880	0.888	
	3	0.939	0.974	0.983	

5 Numerical Examples

Consider the data in Susarla and Van Ryzin (1978). These data represent 81 patients of melanoma. Of them 46 represent whole life time (non-censored data) and the ordered values are: 13, 14, 19, 19, 20, 21, 23, 23, 25, 26, 26, 27, 27, 31, 32, 34, 34, 37, 38, 38, 40, 46, 50, 53, 54, 57, 58, 59, 60, 65, 65, 66, 70, 85, 90, 98, 102, 103, 110, 118, 124, 130, 136, 138, 141, 234.

Using equation (2.12), the value of test statistics, based on the above data is $\hat{\Delta}_{v1_n} = -0.0114$. This value leads to H_o is not rejected at the significance level $\alpha = 0.05$. See Table (3.1). Therefore the data has not DVR Property.

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