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A Simple Integral Representation for Bounded Operators in Topological Vector Spaces

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Abstract. Let E be a locally convex Hausdorff space and let $E^{'}$ be its topological dual, endowed with the weak* topology $\sigma\left(E^{'},E\right)$. Let S be a compact space and let us consider the space $C\left(S,E^{'}\right)$ of all continuous functions $f:S\rightarrow E^{'}$, equipped with the uniform topology. In this paper, we prove a simple integral representation theorem, by means of weak integrals against a scalar measure on S, for a class of linear bounded operators $T:C\left(S,E^{'}\right)$ $\rightarrow E^{'}$. When $E=\Im$ is the Schwartz space on $\mathbb{R}^{n}$ (thus $\Im'$ is the space of tempered distributions), we prove that bounded operators of this class preserve the familiar operations of distribution theory, that is, the operations of derivation and Fourier transform. Also we give an application to weak sequential convergence in this class of operators.

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1. Introduction

Several extensions of the Riesz integral representation Theorem had been proved for various bounded operators on function spaces. See references [2], [3], [7], for the Banach space setting, and reference [4],[10], in the topological vector space context. In this work we consider the case of linear bounded operators $T:C\left(S,E^{'}\right)\rightarrow E^{'}$, where $E^{'}$ is the topological dual of a locally convex vector space E, endowed with the weak* topology $\sigma\left(E^{'},E\right)$, and where $C\left(S,E^{'}\right)$ is the space of continuous functions $f:S\rightarrow E^{'}$ on the compact space S, equipped with the uniform topology, (see below for more details). This consideration comes, in some respects, from the importance of the space $C\left(S,E^{'}\right)$, as it is a basic space in the study of certain stochastic processes describing dynamics of infinite particle systems, see e.g [6], [8]. Then, extending the method used by the author in [7], we show a weak integral representation theorem of the Riesz type for linear bounded operators $T : C\left(S,E^{'}\right) \to E^{'}$, satisfying some mild condition. This theorem is simple compared to those obtained elsewhere with respect to operator valued measures([4],[10]). In the present setting the representation is realized by an integral against a bounded scalar measure on S of the following type:

$$\forall f\in C\left(S,E^{'}\right),\ \forall\xi\in E: \quad \langle Tf,\xi\rangle = \int_S \langle f\left(s\right),\xi\rangle\ \mu\left(ds\right)$$

where μ is a bounded scalar measure which will be attached to the operator T.

When $E=\Im$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, then $\Im'$ is the space of tempered distributions, and we prove that bounded operators of the preceding integral form preserve the familiar operations of distribution theory, that is, the operations of derivation and Fourier transform.

Finally, we give a simple criterion of weak convergence for a sequence T_n of representable operators, using their corresponding measures μ_n.

First let us make precise some notations and facts we shall use in the sequel. E will be a locally convex Hausdorff space and E' its topological dual that is the space of continuous functionals on E. We shall denote the elements of E' by $f,\ g,\ ...$,and refer to the functional duality between E and E' by the symbol:

$$f\in E',\ \xi\in E, \qquad \langle f,\xi\rangle \tag{1}$$

that is the value of the functional f at the vector ξ.

It will be given to the space E' the weak* topology $\sigma\left(E',E\right)$, that is the topology induced by the family of seminorms of the form:

$$p\left(f\right)=p\left(f;x_1,x_2,...,x_n\right)=\underset{1\le j\le n}{Sup}\left|\langle f,x_j\rangle\right| \tag{2}$$

where $\{x_1,x_2,...,x_n\}$ is an arbitrary finite system of elements in E.

On the other hand we consider on the space $C(S,E')$ of continuous functions $f:S\to E^{'}$ the family of seminorms

$$f\in C\left(S,E'\right), \qquad \widetilde{p}\left(f\right)=\underset{t\in S}{Sup}\ p\left(f\left(t\right)\right) \tag{3}$$

where p is given by (2).

Finally the space $C(S)$ of scalar continuous functions on S will be equipped with the uniform norm:

$$f \in C(S)\ ,\ \ \|f\|_\infty = \underset{t\in S}{Sup}\ |f(t)| \tag{4}$$

The following proposition is well known. It gives a natural identification between $(E',\ \sigma(E',E))$ and E :

1.1. **Proposition:** *Every continuous linear functional on E' is of the form*

$$f \to \langle f, \xi \rangle \tag{5}$$

for some (unique) ξ in E.

Also we shall need the following:

1.2. **Proposition:** *(a) The transformations $f \to \langle f(\bullet), \xi\rangle$ of $C(S,E')$ into $C(S)$ are bounded.*
(b) For each $\xi \neq 0$ in E, the transformation $f \to \langle f(\bullet), \xi\rangle$ is onto.

Proof. (a) Indeed, by formulas (3) and (4) we have:
$\|\langle f(\bullet), \xi\rangle\|_\infty = \underset{t\in S}{Sup} |\langle f(t), \xi\rangle| = \widetilde{p}(f)$, where we use (3) with $p(f(t)) = |\langle f(t), \xi\rangle|$.
(b) Since $\xi \neq 0$, there exists a seminorm p on E such that $p(\xi) \neq 0$ and then by the Hahn-Banach Theorem there is an $\alpha' \in E'$ with $\alpha'(\xi) = p(\xi)$. It is clear that we may assume $\alpha'(\xi) = 1$. Now let $h \in C(S)$ and define $f : S \to E'$ by $f(t) = h(t)\alpha'$; then $f \in C(S,E')$ and we have $\langle f(\bullet), \xi\rangle = \langle h(\bullet)\alpha', \xi\rangle = h(\bullet)$. ∎

2. Weak integrals and bounded operators

In this section we will be concerned with bounded $E'-$valued operators on $C(S,E')$, to seek conditions under which such operators define a weak integral on $C(S,E')$ with respect to a scalar measure μ on $S, \mathcal{B}_S$. As we will see, a simple condition does exist and is necessary and sufficient for a bounded operator $T : C(S,E') \to E'$ to have a weak integral form. First, let us make precise some notations and facts.

2.1. **Definition.** *Let $T : C(S,E') \to E'$ be a linear operator. We say that T defines a weak integral on $C(S,E')$ if there exists a bounded signed measure μ on the family $\mathcal{B}_S$ of Borel sets of S such that, for every $\xi \in E$, we have:*

$$\forall f \in C(S,E'), \qquad \langle Tf, \xi\rangle = \int_S \langle f(t), \xi\rangle \quad d\mu(t) \tag{6}$$

In this case we call the vector Tf the weak integral of f.
For an elaborate construction of the weak integral and its properties, see references [1] or [5] . A nice construction is given in [9 p.76].
The preceding definition immediately implies the following:

2.2. **Proposition.** *Let $T : C(S, E') \to E'$ be a linear bounded operator and assume that T defines a weak integral then we have: For ξ, η in E and f, g in $C(S, E')$ we have: $\langle f(\bullet), \xi\rangle = \langle g(\bullet), \eta\rangle \Rightarrow \langle Tf, \xi\rangle = \langle Tg, \eta\rangle$.*

Proof. Obvious from (6) and scalar integration. ∎

Although the above condition has a simple appearance, it turns out that it will be sufficient for a bounded operator $T : C(S, E') \to E'$ to define a weak integral with respect to a signed measure μ on $S, \mathcal{B}_S$.

2.3. **Definition.** *Let $\mathcal{L}$ be the vector space of all bounded operators $T : C(S, E') \to E'$. We define the class $\mathcal{P}_E$ as the class of those operators T in $\mathcal{L}$ which satisfy the following condition:*

(P)
$$if\ \xi, \eta \in E, f, g \in C(S, E')\ then:\ \langle f(\bullet), \xi\rangle = \langle g(\bullet), \eta\rangle \Rightarrow \langle Tf, \xi\rangle = \langle Tg, \eta\rangle$$

2.4. **Proposition.** *If the space $\mathcal{L}$ is equipped with the simple convergence topology then the class $\mathcal{P}_E$ is a closed subspace of $\mathcal{L}$.*

Proof. Immediate. ∎

We are now in a position to give the main result of the paper.

2.5. **Theorem.** For each operator T in the class $\mathcal{P}_E$, there exists a unique signed bounded measure μ on $S, \mathcal{B}_S$ such that $\langle f(\bullet), \xi\rangle$ is $\mu-$integrable for every f *in* $C(S, E')$ and ξ in E, itand we have $\langle Tf, \xi\rangle = \int_S \langle f(t), \xi\rangle\, d\mu(t)$.

Proof. We construct an intermediate bounded operator V, with scalar values, acting on $C(S)$ such that $V\langle f(\bullet), \xi\rangle = \langle Tf, \xi\rangle$ for every $\xi \in E$. This is easy by appealing to condition (P) of T. Let $h \in C(S)$ and $\xi \in E, \xi \neq 0$; by proposition 1-2(b) there exists an $f \in C(S, E')$ such that $\langle f(\bullet), \xi\rangle = h(\bullet)$. Now let us put $Vh = \langle Tf, \xi\rangle$. V is well defined since by condition (P) if $\langle f(\bullet), \xi\rangle = \langle g(\bullet), \eta\rangle = h$ then we have $\langle Tf, \xi\rangle = \langle Tg, \eta\rangle$. It is clear that V is linear and satisfies,

$$\forall \xi \in E : V\langle f(\bullet), \xi\rangle = \langle Tf, \xi\rangle \tag{7}$$

We must show that V is bounded. Let (h_n) be a sequence in $C(S)$ such that $\|h_n\|_\infty \to 0, n \to \infty$. For $\xi \neq 0$, write $h_n = \langle f_n(\bullet), \xi\rangle$ where $f_n \in C(S, E')$,then $\|h_n\|_\infty = \widetilde{p}(f_n) \to 0$,where .$\widetilde{p}$ is given by (3) . Since T is bounded we deduce that $Tf_n \to 0$ and then $|Vh_n| = |\langle Tf_n, \xi\rangle| \to 0$. Consequently $V \in C(S)^*$, the conjugate space of $C(S)$. By the Riesz representation Theorem, there exists a signed bounded measure μ on $S, \mathcal{B}_S$ such that

$$\forall h \in C(S);\ Vh = \int_s h d\mu \tag{8}$$

By formula (7) we deduce :$V\langle f(\bullet), \xi\rangle = \int_S \langle f, \xi\rangle d\mu = \langle Tf, \xi\rangle$, that is $\int_S \langle f, \xi\rangle d\mu = \langle Tf, \xi\rangle$ for every ξ. But this is formula (6) of definition 2.1, and shows that Tf is a weak integral as wanted. ∎

2.6. **Example.** Let us fix s in S and consider the bounded operator $T : C(S, E') \to E'$ given by $Tf = f(s)$, $f \in C(S, E')$. Then it is easy to check that T is in the class $\mathcal{P}_E$. Now we compute the scalar measure μ attached to T, according to theorem 2.5. We have $\int_s \langle f, \xi\rangle d\mu = \langle Tf, \xi\rangle = \langle f(s), \xi\rangle$, by the nature of T. But $\langle f(s), \xi\rangle$ is the value of the integral of the function $t \to \langle f(t), \xi\rangle$, with respect to the Dirac measure δ_s on the point s; that is we have $\int_s \langle f(t), \xi\rangle\ \mu(dt) = \int_s \langle f(t), \xi\rangle\ \delta_s(dt)$, for all $f \in C(S, E')$ and all $\xi \in E$. Now take $h \in C(S)$ and $\xi \neq 0$ in E; by proposition1.2(b) there exists $f \in C(S, E')$ such that $\langle f(t), \xi\rangle = h(t)\ \forall t \in S$ and an application of the above integrals gives $\int_s h(t)\, \mu(dt) = \int_s h(t)\, \delta_s(dt)$. Since h is arbitrary in $C(S)$, this in turn gives $\mu = \delta_s$, by the classical Riesz Theorem.

As a second illustration of theorem 2.5, let us consider the space $C([0,1], \Im')$, where $\Im$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and $\Im'$ the space of tempered distributions (see [11]). To simplify matters we take $n = 1$. Recall that we equipped $\Im'$ with the weak* topology.

First let us define $S' = \frac{dS}{dx}$ as the function $t \to \frac{dS_t}{dx}$, where $\frac{dS_t}{dx}$ is the usual derivative of the distribution S_t. Let us observe that $t \to \frac{dS_t}{dx}$ is a continuous function on $[0,1]$ into $\Im'$, by the convergence criteria in $\Im'$. Thus we put $S' : t \to \frac{dS_t}{dx}$, and we have $S' \in C([0,1], \Im')$.

Likewise by using a similar device, we define the Fourier transform of $S \in C([0,1], \Im')$ by $\hat{S} : t \to \hat{S}_t$, where $\hat{S}_t$ has the meaning of $\hat{S}_t(\xi) = S_t\left(\overset{\curlywedge}{\xi}\right)$, where $\overset{\curlywedge}{\xi}$ is the Fourier transform of the function $\xi \in \Im$. Since $t \to \hat{S}_t$ is continuous from $[0,1]$ into $\Im'$, this defines $\hat{S}$ as an element of $C([0,1], \Im')$. With this data, we have the following:

2.7. **Proposition.** *Suppose that $\Lambda : C([0,1], \Im') \to \Im'$ is an operator in the class $\mathcal{P}_{\Im'}$ of definition 2.3; then we have:*

$$(\Lambda S)^{'} = \Lambda S^{'} \tag{10}$$

and

$$\overset{\wedge}{\Lambda S} = \Lambda \hat{S} \tag{11}$$

Proof. According to theorem 2.5 we have:

$$S \in C([0,1], \Im'),\ \xi \in \Im\colon\ \langle \Lambda S, \xi \rangle = \int\limits_{[0,1]} \langle S_t, \xi \rangle \, d\mu(t) \tag{12}$$

for some unique bounded measure μ on $[0,1]$.

To see formula (10), we perform the following simple computation for ξ in $\Im$: $(\Lambda S)^{'}(\xi) = \frac{d\Lambda S}{dx}(\xi) = -\left\langle \Lambda S, \xi^{'} \right\rangle$, where $\xi^{'} = \frac{d\xi}{dx}$. Then, citing (12), we get $-\left\langle \Lambda S, \xi^{'} \right\rangle = -\int\limits_{[0,1]} \left\langle S_t, \xi^{'} \right\rangle d\mu(t) = \int\limits_{[0,1]} \left\langle \frac{dS_t}{dx}, \xi \right\rangle d\mu(t)$, which is exactly $\Lambda S^{'}(\xi)$; since ξ is arbitrary, this proves (10).

To see formula (11), let us compute the Fourier transform of the tempered distribution ΛS. We have:

$\overset{\wedge}{\Lambda S}(\xi) = \left\langle \Lambda S, \overset{\wedge}{\xi} \right\rangle = \int\limits_{[0,1]} \left\langle S_t, \overset{\wedge}{\xi} \right\rangle d\mu(t) = \int\limits_{[0,1]} \left\langle \hat{S}_t, \xi \right\rangle d\mu(t)$; but this last integral is $\Lambda \hat{S}(\xi)$ by (12) and the definition of $\hat{S}$. This proves formula (11). ∎

3. Estimating the $\xi-$norm of T.

In the framework of theorem 2.5, let us define the $\xi-$norm ($\xi \in E$) of a bounded operator $T : C(S, E') \to E'$ by:

$$\|T\|_{\xi} = Sup\ \{|\langle Tf, \xi \rangle| : f \in C(S, E'), \|\langle f(\bullet), \xi \rangle\|_{\infty} \leq 1.\ \} \tag{12}$$

Let us observe that if $g \in E'$ and $\xi \in E$, then $|\langle g, \xi \rangle|$ is the value of the seminorm $|\langle ., \xi \rangle|$ on E' at the vector g. With this ingredients we have:

3.1. **Proposition.** *Under the hypothesis of Theorem 2.5 we have:*

$$\|T\|_{\xi} = v(\mu),\ \xi \in E \tag{13}$$

where $v(\mu)$ is the total variation of the scalar measure μ attached to the bounded operator T.

Proof. Consider the bounded functional V given in the proof of theorem 2.5. Then by the Riesz Theorem we have $\|V\| = v(\mu)$, and citing (7) gives $V\langle f(\bullet), \xi\rangle = \langle Tf, \xi\rangle$, for $\xi \in E$ and $f \in C(S, E')$. It remains to check that:

(14) $$\|V\| = Sup\ \{|\langle Tf, \xi\rangle| :), f \in C(S, E')\ \ \|\langle f(\bullet), \xi\rangle\|_\infty \leq 1.\ \}$$

Denoting by α the right hand side of (14) and taking into account the definition
$\|V\| = Sup\ \{|Vh| : h \in C(S), \|h\|_\infty \leq 1.\ \}$ it is clear that $\alpha \leq \|V\|$. On the other hand, for each $\epsilon > 0$, there is an $h \in C(S)$ such that $\|h\|_\infty \leq 1$ and $\|V\| - \epsilon < |Vh|$. By proposition1.2(b), if $\xi \in E, \xi \neq 0$, there exists $f \in C(S, E')$ such that $\langle f(\bullet), \xi\rangle = h$ and thus $\|\langle f(\bullet), \xi\rangle\|_\infty = \|h\|_\infty \leq 1$. We have also $|V\langle f(\bullet), \xi\rangle| = |Vh| \leq \alpha$,by the nature of α. Consequently, by the choice of h, we get $\|V\| - \epsilon < |Vh| \leq \alpha$ and since $\epsilon > 0$ is arbitrary, we obtain $\|V\| \leq \alpha$ which gives the equality $\|V\| = \alpha$ and then, in view of (12), the validity of (13). ∎

4. An Application.

4.1. **Theorem.** *Let T_n be a sequence of bounded operators in the class $\mathcal{P}_E$ and let $T : C(S, E') \to E'$ be bounded. If T_n converges weakly to the operator T, then T is in the class $\mathcal{P}_E$. Moreover, assume that μ_n and μ are respectively the corresponding measures of T_n and T according to Theorem 2.5, then we have:*

$$\forall h \in C(S), \quad \underset{n}{Lim} \int_S h\, d\mu_n = \int_S h\, d\mu.$$

This means that the sequence of bounded measures μ_n converges weakly to the bounded measure μ. On the other hand if μ_n converges weakly to μ then T_n converges weakly to T.

Proof. The weak convergence of T_n means $\underset{n}{Lim}\langle T_n f, \xi\rangle = \langle Tf, \xi\rangle$, for all f in $C(S, E')$ and all ξ in E, (see Proposition 1.1). From this it follows easily that T is in the class $\mathcal{P}_E$. Now by Theorem 2.5, the preceding limit implies $\underset{n}{Lim} \int_s \langle f(s), \xi\rangle\ d\mu_n(s) = \int_S \langle f(s), \xi\rangle\ d\mu$, for all f in $C(S, E')$ and all ξ in E. If $h \in C(S)$, and $\xi \in E$, there exists by proposition1.2(b), an f in $C(S, E')$ such that $h(s) = \langle f(s), \xi\rangle\ \forall s \in S$; from this we deduce that $\underset{n}{Lim} \int_S h\, d\mu_n = \int_S h\, d\mu$ as wanted. The converse is clear. ∎

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