Applied Mathematical Sciences, Vol. 3, 2009, no. 18, 869 - 876

Classification of Spatially Homogeneous Rotating Space-Times According to their Conformal Vector Fields

Ghulam Shabbir and Amjad Ali

Faculty of Engineering Sciences,
GIK Institute of Engineering Sciences and Technology
Topi Swabi, NWFP, Pakistan.
shabbir@giki.edu.pk

Abstract

Direct integration technique is used to study non conformally flat spatially homogeneous rotating space-times according to their conformal vector fields. It is shown that the above space-times do not admit proper conformal vector fields. Conformal vector fields for the above space-times are homothetic vector fields. Here we also discuss some well known examples of spatially homogeneous rotating space-times according to their conformal vector fields.

Keywords: Direct integration technique; Conformal vector field; Homothetic vector fields.

1. Introduction

There has been much interest in studying symmetry in general relativity but conformal symmetry is ignored due to the lack of linearity property [1-4]. The conformal symmetry which preserves the space-time structure up to a conformal factor carries significant information and plays an important role in Einstein's theory of general relativity. It is therefore important to study conformal symmetry. In this paper we investigate the conformal symmetry of non conformally flat spatially homogeneous rotating space-times using direct integration technique. Here we also discuss the conformal symmetry of some well know spatially homogeneous rotating space-times like; Reboucas, Som-Raychaudhuri, Hoenselaers-Vishveshwara, Godel-Friedmann and Stationary Godel space-times.

Throughout M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^{a}{}_{bcd}$, the Ricci tensor components are $R_{ab} = R^{c}{}_{acb}$ and the Weyl tensor components are $C^{a}{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time M will be assumed non conformally flat in the sense that the Weyl tensor does not vanish over any non empty open subset of M.
The covariant derivative of any vector field X on M can be decomposed as

$$X_{a;b} = \frac{1}{2} h_{ab} + F_{ab}, \tag{1}$$

where $h_{ab} (= h_{ba}) = L_X g_{ab}$ and $F_{ab} (= -F_{ba})$ are symmetric and skew symmetric tensors on M respectively. Such a vector field X is called conformal if the local diffeomorphisms ϕ_t (for appropriate t) associated with X preserve the metric structure up to a conformal factor i.e $\phi_t^{*} g_{ab} = \psi\, g_{ab}$, where ψ is a no where zero smooth function on M, called the conformal function of X and ϕ_t^{*} is a pulback map on M [6]. This is equivalent to the condition that h_{ab} satisfies

$h_{ab} = 2\psi\, g_{ab}$,

which in turn equievalent to

$$g_{ab,c} X^{c} + g_{bc} X^{c}_{,a} + g_{ca} X^{c}_{,b} = 2\psi\, g_{ab}, \tag{2}$$

for some smooth conformal function ψ on M, then X is called a conformal vector field. If ψ is some constant on M, X is homothetic (proper homothetic if $\psi \neq 0$) while $\psi = 0$ it is Killing. The vector field X is called proper conformal if it is not homothetic while it is special conformal if the conformal function ψ satisfies the condition $\psi_{a;b} = 0$.

2. Main Results

Consider spatially homogeneous rotating space-times with line element in the usual coordinate system (t, r, ϕ, z) (labeled by (x^0, x^1, x^2, x^3), respectively) given by [5]

$$ds^2 = -dt^2 + dr^2 + A(r) d\phi^2 + dz^2 + 2\, B(r) dt\, d\phi, \tag{3}$$

where A and B are no where zero functions of r only. The above space-times (3) admit at least three linearly independent Killing vector fields which are $\frac{\partial}{\partial t}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}$. A vector field X is called conformal vector field if it satisfies equation (2). Writing (2) explicitly and using equation (3) we get

$$X^0_{,0} - B(r)X^2_{,0} = \psi(t,r,\phi,z), \tag{4}$$

$$X^1_{,0} - X^0_{,1} + B(r)X^2_{,1} = 0, \tag{5}$$

$$\frac{B_r(r)}{2B(r)}X^1 + \frac{1}{2}X^0_{,0} - \frac{1}{2B(r)}X^0_{,2} + \frac{A(r)}{2B(r)}X^2_{,0} + \frac{1}{2}X^2_{,2} = \psi(t,r,\phi,z), \tag{6}$$

$$X^3_{,0} - X^0_{,3} + B(r)X^2_{,3} = 0, \tag{7}$$

$$X^1_{,1} = \psi(t,x,y,z), \tag{8}$$

$$B(r)X^0_{,1} + A(r)X^2_{,1} + X^1_{,2} = 0, \tag{9}$$

$$X^3_{,1} + X^1_{,3} = 0, \tag{10}$$

$$\frac{A_r(r)}{2A(r)}X^1 + \frac{B(r)}{A(r)}X^0_{,2} + X^2_{,2} = \psi(t,x,y,z), \tag{11}$$

$$X^3_{,2} + B(r)X^0_{,3} + A(r)X^2_{,3} = 0 \tag{12}$$

$$X^3_{,3} = \psi(t,x,y,z). \tag{13}$$

Equations (8) and (13) give

$$X^1 = \int \psi(t,r,\phi,z)dr + E^1(t,\phi,z), \quad X^3 = \int \psi(t,r,\phi,z)dz + E^2(t,r,\phi).$$

Equations (4), (5) and (7) give

$$X^2 = \frac{1}{B_r(r)}\left[E^1_t(t,\phi,z) - zE^2_{tr}(t,r,\phi) - \frac{1}{2}z^2\psi_{tr}(t,r,\phi,z)\right] + E^3(t,r,\phi),$$

$$X^0 = \frac{B(r)}{B_r(r)}\left[E^1_t(t,\phi,z) - zE^2_{tr}(t,r,\phi) - \frac{1}{2}z^2\psi_{tr}(t,r,\phi,z)\right] + \\ + B(r)E^3(t,r,\phi) + \int \psi(t,r,\phi,z)dt + E^4(r,\phi,z),$$

Thus we have the following system

$$\begin{aligned}
X^0 &= \frac{B(r)}{B_r(r)}\left[E^1_t(t,\phi,z) - zE^2_{tr}(t,r,\phi) - \frac{1}{2}z^2\psi_{tr}(t,r,\phi,z)\right] + \\
&\quad + B(r)E^3(t,r,\phi) + \int \psi(t,r,\phi,z)dt + E^4(r,\phi,z), \\
X^1 &= \int \psi(t,r,\phi,z)dr + E^1(t,\phi,z), \\
X^2 &= \frac{1}{B_r(r)}\left[E^1_t(t,\phi,z) - zE^2_{tr}(t,r,\phi) - \frac{1}{2}z^2\psi_{tr}(t,r,\phi,z)\right] + E^3(t,r,\phi), \\
X^3 &= \int \psi(t,r,\phi,z)dz + E^2(t,r,\phi),
\end{aligned} \tag{14}$$

where $E^1(t,\phi,z), E^2(t,r,\phi), E^3(t,r,\phi), E^4(r,\phi,z)$ are functions of integration which will be determined by solving the remaining equations. If one proceeds further after doing some hard and lengthy calculation we get $E^1(t,\phi,z)=F^1(t,\phi)$, $E^2(t,r,\phi)=c_1, \psi=c_2$ and $E^4(r,\phi,z)=F^2(r,\phi)$, where $F^1(t,\phi)$ and $F^2(r,\phi)$ are functions of integration and $c_1, c_2 \in R$. Here we have $\psi=c_2$, which means that the above space-times do not admit proper conformal vector fields. Now we are looking for homothetic vector fields of the above space-times if they exist. Substituting the above information in (14), we get the following system

$$\begin{aligned} &X^0=\frac{B(r)}{B_r(r)}F_t^1(t,\phi)+B(r)E^3(t,r,\phi)+F^2(r,\phi)+c_2 t, \qquad X^1=F^1(t,\phi)+c_2 r,\\ &X^2=\frac{1}{B_r(r)}F_t^1(t,\phi)+E^3(t,r,\phi), \qquad X^3=c_1+c_2 z. \end{aligned} \tag{15}$$

Differentiate (5) with respect to t and using (15) we get $E^3(t,r,\phi)=F^3(r,\phi)$, where $F^3(r,\phi)$ is a function of integration. Substituting this information in (15) we have

$$\begin{aligned} &X^0=\frac{B(r)}{B_r(r)}F_t^1(t,\phi)+B(r)F^3(r,\phi)+F^2(r,\phi)+c_2 t, \qquad X^1=F^1(t,\phi)+c_2 r,\\ &X^2=\frac{1}{B_r(r)}F_t^1(t,\phi)+F^3(r,\phi), \qquad X^3=c_1+c_2 z. \end{aligned} \tag{16}$$

Now differentiating (11) with respect to t and r and using (16) we get

$$\left[\frac{A_r B_r}{A+B^2}\right]' F_t^1(t,\phi)=0 \tag{17}$$

Equation (17) gives the following three possibilities:

I: $F_t^1(t,\phi)=0$ and $\left[\frac{A_r B_r}{A+B^2}\right]' \neq 0.$ **II:** $F_t^1(t,\phi)\neq 0$ and $\left[\frac{A_r B_r}{A+B^2}\right]' = 0.$

III: $F_t^1(t,\phi)=0$ and $\left[\frac{A_r B_r}{A+B^2}\right]' = 0.$

We will consider each case in turn.

Case I: In this case we have $F_t^1(t,\phi)=0$ and $\left[\frac{A_r B_r}{A+B^2}\right]' \neq 0.$ Equation $F_t^1(t,\phi)=0 \Rightarrow$ $F^1(t,\phi)=G^1(\phi)$, where $G^1(\phi)$ is a function of integration. Thus (16) becomes

$$\begin{aligned} &X^0=B(r)F^3(r,\phi)+F^2(r,\phi)+c_2 t, \qquad X^1=G^1(\phi)+c_2 r,\\ &X^2=F^3(r,\phi), \qquad X^3=c_1+c_2 z. \end{aligned} \tag{18}$$

Differentiate (5) and (6) with respect to ϕ, using (18) and then substituting back in (9) and (11), we have the following information

$F^2(r,\phi)=c_3\,\phi\,B_r(r)+\phi(rB_r-B)-c_4B+c_5,\ B(r)=\dfrac{c_6}{c_2-c_7}(c_3+c_2r)^{\frac{c_2-c_7}{c_2}},$

$A(r)=c_8(c_3+c_2r)^{2(\frac{c_2-c_7}{c_2})},\ F^3(r,\phi)=c_7\phi+c_8,$ where $c_2,c_3,c_4,c_5,c_6,c_7,c_8\in R$ $(c_2\neq c_7, c_2\neq 0).$ Thus (18) becomes

$$X^0=c_2t+c_5,\ X^1=c_2r+c_3,\ X^2=c_7\phi+c_4,\ X^3=c_2z+c_1. \tag{19}$$

The space-time in this case takes the form

$$ds^2=-dt^2+dr^2+c_8(c_3+c_2r)^{2(\frac{c_2-c_7}{c_2})}d\phi^2+dz^2+2\,\frac{c_6}{c_2-c_7}(c_3+c_2r)^{\frac{c_2-c_7}{c_2}}\,dt\,d\phi.$$

The above space-time admits four homothetic vector fields in which three are Killing vector fields and one is proper homothetic vector field which is

$$X=(c_2t,\ c_2r+c_3,\ c_7\,\phi,\ c_2z). \tag{20}$$

Case II: In this case we have $F_t^1(t,\phi)\neq 0$ and $\left[\dfrac{A_rB_r}{A+B^2}\right]'=0$ which implies that $\dfrac{A_rB_r}{A+B^2}=c_9,$ where $c_9\in R.$ If one proceeds further after some lengthy calculation one finds that $F_t^1(t,\phi)=0$ which gives contradiction to our assumption (here we assume that $F_t^1(t,\phi)\neq 0$). Hence this case does not exist.

Case III: In this case we have $F_t^1(t,\phi)=0$ and $\left[\dfrac{A_rB_r}{A+B^2}\right]'=0.$ Equation $\left[\dfrac{A_rB_r}{A+B^2}\right]'=0\Rightarrow$ $\dfrac{A_rB_r}{A+B^2}=a,$ where $a\in R.$ Here there exist two possibilities:

(P) $a\neq 0$ (Q) $a=0$

In (P) if one proceeds further one finds that $A_r(r)=0$ which implies that $a=0$ which gives contradiction to our assumption that $a\neq 0.$ Hence this sub case is not possible.
In (Q) there exist further two sub cases

(i) $A_r(r)=0,\ B_r(r)\neq 0.$ (ii) $A_r(r)\neq 0\ \ B_r(r)=0.$

In (i) if we proceed further we find that $B(r)=$ constant which gives contradiction to our assumption that $B_r(r)\neq 0.$ Hence this sub case is not possible.
In (ii) we have $A_r(r)\neq 0$ and $B_r(r)=0.$ Equation $B_r(r)=0\Rightarrow B(r)=b,$ where $b\in R.$
If one proceeds further one finds the solution of the equations from (4) to (13) as follows

$$\begin{aligned}&X^0=b\left[\frac{c_{11}}{c_{13}c_{14}}e^{-c_{13}r}-\frac{1}{4}c_{11}c_{13}\,\phi^2-\frac{1}{2}c_{12}\,c_{13}\,\phi\right]+c_{16}, \qquad X^1=c_{11}\,\phi+c_{12},\\ &X^2=\frac{c_{11}}{c_{13}c_{14}}e^{-c_{13}r}-\frac{1}{4}c_{11}c_{13}\,\phi^2-\frac{1}{2}c_{12}\,c_{13}\,\phi+c_{15}, \qquad X^3=c_{17}.\end{aligned} \tag{21}$$

where $c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17} \in R (c_{13} \neq 0, c_{14} \neq 0)$. Here $A(r) = \left(c_{14}\, e^{c_{13} r} - b^2\right)$. In this case conformal vector fields are Killing vector fields.

3. Examples

In this section we will discuss conformal vector fields of some spatially homogeneous space-times. Here we will only present the results and calculation will be omitted. One can easily reproduce the following results by using the general method, which is given in section 2. These examples are as follows:

(1) **Reboucas space-time**

Here if we choose $A(r) = -(1+3\cosh^2 2r)$ and $B(r) = 2\cosh 2r$, the above space-time (3) becomes Reboucas space-time and takes the form [5]

$$ds^2 = -dt^2 + dr^2 - (1+3\cosh^2 2r) d\phi^2 + dz^2 + 4\cosh 2r\, dt\, d\phi. \tag{22}$$

For the above space-time conformal vector fields are

$$\begin{aligned} X^0 &= -2\operatorname{cosech} 2r \left[c_{33}\sin 2\phi - c_{34}\cos 2\phi\right] + c_{35}, \\ X^1 &= c_{33}\cos 2\phi + c_{34}\sin 2\phi, \\ X^2 &= -\coth 2r \left[c_{33}\sin 2\phi - c_{34}\cos 2\phi\right] + c_{36},\ X^3 = c_{37}, \end{aligned} \tag{23}$$

where $c_{33}, c_{34}, c_{35}, c_{36}, c_{37} \in R.$ Here conformal vector fields are Killing vector fields.

(2) **Som-Raychaudhuri space-time**

In this case if we choose $A(r) = r^2(1-r^2)$ and $B(r) = r^2$ the above space-time (3) becomes Som-Raychaudhuri space-time and takes the form [5]

$$ds^2 = -dt^2 + dr^2 + r^2(1-r^2) d\phi^2 + dz^2 + 2r^2\, dt\, d\phi. \tag{24}$$

For the above space-time conformal vector fields are

$$\begin{aligned} & X^0 = r\left[c_{38}\sin\phi - c_{39}\cos\phi\right] + c_{40},\ X^1 = c_{38}\cos\phi + c_{39}\sin\phi, \\ & X^2 = -\frac{1}{r}\left[c_{38}\sin\phi - c_{39}\cos\phi\right] + c_{41},\ X^3 = c_{42}, \end{aligned} \tag{25}$$

where $c_{38}, c_{39}, c_{40}, c_{41}, c_{42} \in R.$ In this case conformal vector fields are Killing vector fields.

(3) **Hoenselaers-Vishveshwara space-time**

Here if we choose $A(r) = -\frac{1}{2}(\cosh r - 1)(\cosh r - 3)$ and $B(r) = (\cosh r - 1)$ the above space-time (3) becomes Hoenselaers-Vishveshwara space-time and takes the form [5]

$$ds^2 = -dt^2 + dr^2 - \frac{1}{2}(\cosh r - 1)(\cosh r - 3) d\phi^2 + dz^2 + 2(\cosh r - 1)\, dt\, d\phi. \tag{26}$$

For the above space-time conformal vector fields are

$$X^0 = \sqrt{2}\left(\coth r - \cos echr\right)\left[c_{43}\sin\frac{\phi}{\sqrt{2}} - c_{44}\cos\frac{\phi}{\sqrt{2}}\right] + c_{45},$$
$$X^1 = c_{43}\cos\frac{\phi}{\sqrt{2}} + c_{44}\sin\frac{\phi}{\sqrt{2}}, \tag{27}$$
$$X^2 = -\sqrt{2}\coth r\left[c_{43}\sin\frac{\phi}{\sqrt{2}} - c_{44}\cos\frac{\phi}{\sqrt{2}}\right] + c_{46},\ X^3 = c_{47},$$

where $c_{43}, c_{44}, c_{45}, c_{46}, c_{47} \in R.$ Here conformal vector fields are Killing vector fields.

(4) **Godel-Friedmann space-time:**

In this case if we choose $A(r) = \sinh^2 r(1-\sinh^2 r)$ and $B(r) = \sqrt{2}\sinh^2 r$ the above space-time (3) becomes Godel-Friedmann space-time and takes the form

$$ds^2 = -dt^2 + dr^2 + \sinh^2 r(1-\sinh^2 r)d\phi^2 + dz^2 + 2\sqrt{2}\sinh^2 r\, dt\, d\phi. \tag{28}$$

For the above space-time conformal vector fields are

$$X^0 = \sqrt{2}\left[\sinh 2r - \tanh r\cosh 2r\right]\left[c_{48}\sin\phi - c_{49}\cos\phi\right] + c_{50},$$
$$X^1 = c_{48}\cos\phi + c_{49}\sin\phi, \tag{29}$$
$$X^2 = -2\coth 2r\left[c_{48}\sin\phi - c_{49}\cos\phi\right] + c_{51},\ X^3 = c_{52},$$

where $c_{48}, c_{49}, c_{50}, c_{51}, c_{52} \in R.$ In this case conformal vector fields are Killing vector fields.

(5) **Stationary Godel Space-time:**

Here, if we choose $A(r) = -\frac{1}{2}e^{2ar}$ and $B(r) = e^{ar}$, where $a \in R\setminus\{0\}.$ The above space-time (3) becomes stationary Godel space-time and takes the form

$$ds^2 = -dt^2 + dr^2 - \frac{1}{2}e^{2ar}d\phi^2 + dz^2 + 2e^{ar}\, dt\, d\phi. \tag{30}$$

For the above space-time conformal vector fields are

$$X^0 = \frac{2}{a}c_{53}\, e^{-ar} + c_{54},\ X^1 = c_{53}\,\phi + c_{55},$$
$$X^2 = -c_{53}\left[\frac{a}{2}\phi^2 + \frac{2}{a}e^{-ar}\sinh ar\right] - c_{55}\, a\phi + c_{57},\ X^3 = c_{56}, \tag{31}$$

where $c_{53}, c_{54}, c_{55}, c_{56}, c_{56}, c_{57} \in R.$ In this case conformal vector fields are Killing vector fields.

References

[1] R. F. Bilyalov, conformal transformation groups in gravitational fields, Sov. Physics, **8** (1964), 878.

[2] L. Defrise-Carter, conformal groups and conformally equivalent isometry groups, Commun. Math. Physics, **40** (1975), 273-282.

[3] Michael Steller, conformal vector fields in space-time, Ann. Glob. Annal Geo, **29** (2006), 293-317.

[4] G. S. Hall and J. D. Steele, conformal vector fields in general relativity, J. Math. Physics, **32** (1991), 1847-1853.

[5] K. D. Krori, P. Borgohain, P. K. Kar and Dipali Das (Kar), Exact scalar and spinor solutions in some rotating universes, J. Math. Physics, **29** (1988), 1645-1649.

[6] R. M. Wald, General Relativity, the University of Chicago Press, 1984.

Received: October, 2008