Applied Mathematical Sciences, Vol. 3, 2009, no. 18, 877 - 881

Vertex, Edge and Total Coloring in Spider Graphs

Sadegh Rahimi Sharebaf

Department of Mathematics
Shahrood University of Technology, Shahrood, Iran
srahimi@shahroodut.ac.ir

Abstract

In this paper we investigate the vertex chromatic number, the edge chromatic number, and the total chromatic number in Spider graphs.

Mathematical Subject Classification: 05xx

Keywords: Vertex coloring, Edge- coloring, Total coloring

1 Introduction

A vertex coloring of a graph G is a labeling $f : V(G) \to T$, where T is a non-empty set. We usually refer T as the set of colors. f is proper vertex coloring if the adjacent vertices have different labels. A graph G is k-vertex colorable if it has a proper vertex coloring $f : V(G) \to T$ and $|T| = k$. The vertex chromatic number $\chi(G)$ is the least k such that G is k-vertex colorable. An edge coloring of a graph G is a labeling $f : E(G) \to T$, where T is a non-empty set. We usually refer T as the set of colors. f is proper edge coloring if the adjacent edges have different labels. A graph G is k-edge colorable if it has a proper edge coloring $f : E(G) \to T$ and $|T| = k$. The edge chromatic number $\chi'(G)$ is the least k such that G is k-edge colorable, [3].

A total coloring of a graph G is a labeling $f : V(G) \cup E(G) \to T$, where T is a non-empty set. We usually refer T as the set of colors. f is proper total coloring if not only the adjacent vertices, and the adjacent edges have different labels, but also the label of any edge is different from the labels of its two end-points. A graph G is k-total colorable if it has a proper total coloring $f : V(G) \cup E(G) \to T$ and $|T| = k$. The total chromatic number $\chi_t(G)$ is the least k such that G is k-total colorable, [4].

It is well known that $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum vertex degree. Vizing theorem asserts that $\chi'(G) \leq 1 + \Delta(G)$.

Clearly, for any graph G, $\chi_t(G) \geq 1+\Delta(G)$. The Behzad-Vizing conjecture claims that $\chi_t(G) \leq 2 + \Delta(G)$. This conjecture has benn verified for several classes of graphs,[1,4,5].

A Spider is a tree with at most one vertex of degree more than two, called the center of Spider (if no vertex of degree more than two, then any vertex can be the center). A leg of a Spider is a path from the center to a vertex of degree one. Thus, a star with k legs is a Spider of k legs, each of lenght 1, [2].

Let $m \geq 3$ be an integer. We define the Spider graph S_m as the graph obtained from a Spider T with m legs $P_1, P_2, ..., P_m$ each of length at least two, such that two vertex x, y of two different legs P_i and P_j are adjacent if $|i - j| \in \{1, m - 1\}$, and $d(x, o) = d(y, o)$ where o is the center of T.

In this paper we consider a subclass of Spider graphs. Let S_m be a spider graph with legs $P_1, P_2, ..., P_m$. If any leg P_i has length n for $i = 1, 2, ..., m$, and for any two vertices x, y of two different legs P_i and P_j with $|i-j| \in \{1, m-1\}$, we have v_i is adjacent to v_j, then we call the Spider graph, a complete Spider graph, and denote it by $S_{m,n}$. In this paper we study the vertex coloring, the edge-coloring and the total coloring in complete Spider graphs.

2 Results

Theorem 1 *$\chi(S_{m,n}) = 3$ if m is even, and $\chi(S_{m,n}) = 4$ if m is odd.*

Proof. Since $S_{m,n}$ has triangles, we have $\chi(S_{m,n}) \geq 3$. Also if m is odd, then the nearest cycle to the center is an odd cycle and therefore has chromatic number 3. But the center is adjacent to all of those vertices. So $\chi(S_{m,n}) \geq 4$. Let $V(P_i) = \{o, v_{i1}, ..., v_{in}\}$. Let C be the nearest cycle to the center.

If m is even, then we consider a 2-coloring $f : V(C) \to T$, where $T = \{1, 2\}$. We color the center by 3. For any vertex x in leg P_i if $d(x, v_{i1})$ is even we color x by the color of v_{i1}, and if $d(x, v_{i1})$ is odd we color x by the color of $v_{(i+1)1}$. So $\chi(S_{m,n}) = 3$.

If m is odd, then we consider a 3-coloring $f : V(C) \to T$, where $T = \{1, 2, 3\}$. We color the center by 4. For any vertex x in leg P_i if $d(x, v_{i1})$ is even we color x by the color of v_{i1}, and if $d(x, v_{i1})$ is odd we color x by the color of $v_{(i+1)1}$. So $\chi(S_{m,n}) = 4$. ■

Theorem 2 $\chi'(S_{m,n}) = \Delta(G)$.

Proof. First $\chi'(S_{m,n}) \geq \Delta(G)$. Let o be the center and let $V(P_i) = \{o, v_{i1}, ..., v_{in}\}$ for $i = 1, 2, ..., m$. We define an edge coloring as follows:

$f(ov_{i1}) = i$ for $i = 1, 2, ..., m$,

$f(v_{i1}v_{(i+1)1}) = f(ov_{i+2})$, for $i = 1, 2, ..., m$, where the addition is calculated in modulo m,

$f(v_{ik}v_{(i+1)k}) = f(v_{i1}v_{(i+1)1})$ for $i = 1, 2, ..., m$, and $k = 2, 3, ..., n$.

Case 1. If $m \geq 4$, then $f(v_{i1}v_{i2}) = f(ov_{(i-1)1})$ for $i = 1, 2, ..., m$,

$f(v_{ik}v_{i(k+1)}) = f(ov_{i1})$ if k is odd, and $f(v_{ik}v_{i(k+1)}) = f(v_{i1}v_{i2})$ if k is even.

Case 2. If $m = 3$, then $f(v_{i1}v_{i2}) = 4$ for $i = 1, 2, ..., m$,

$f(v_{ik}v_{i(k+1)}) = 4$ if k is odd, and $f(v_{ik}v_{i(k+1)}) = f(ov_{i1})$ if k is even.

So we use Δ colors, and then $\chi'(S_{m,n}) \leq \Delta(G)$. Hence $\chi'(S_{m,n}) = \Delta(G)$. ■

Theorem 3 *For* $m = 3, 4$, $\chi_t(S_{m,n}) = \Delta(G) + 2$.

Proof. Let $m = 3$. First we notice that $\chi_t(S_{3,n}) \geq 6$. We define a total coloring as follows: $f(ov_{i1}) = i$ for $i = 1, 2, 3$, $f(v_{11}v_{21}) = f(v_{1k}v_{2k}) = 4$,$f(v_{21}v_{31}) = f(v_{2k}v_{3k}) = 5$, $f(v_{31}v_{11}) = f(v_{3k}v_{1k}) = 6$, for $k = 1, 2, .., n$,

$f(v_{1k}v_{1(k+1)}) = 3$ if k is odd, $f(v_{1k}v_{1(k+1)}) = 1$ if k is even,

$f(v_{2k}v_{2(k+1)}) = 1$ if k is odd, $f(v_{2k}v_{2(k+1)}) = 2$ if k is even,

$f(v_{3k}v_{3(k+1)}) = 2$ if k is odd, $f(v_{2k}v_{2(k+1)}) = 3$ if k is even,

$f(o) = 5$, $f(v_{11}) = 2$, $f(v_{21}) = 3$, $f(v_{31}) = 1$, $f(v_{12}) = 5$, $f(v_{22}) = 6$, $f(v_{32}) = 4$,

$f(v_{ik}) = f(v_{i1})$ if k is odd, $f(v_{ik}) = f(v_{i2})$ if k is even. Since we use 6 colors, then $\chi_t(S_{3,n}) \leq 6$.

For $m = 4$, we first notice that $\chi_t(S_{4,n}) \geq 6$. Now we define a total coloring as follows:

$f(ov_{i1}) = i$ for $i = 1, 2, 3, 4$,

$f(v_{i1}v_{i2}) = f(ov_{(i+1)1})$,

For $k = 1, 2, ..., n$, $f(v_{ik}v_{(i+1)k}) = 5$, if i is odd, $f(v_{ik}v_{(i+1)k}) = 6$, if i is even,

$f(v_{ik}v_{i(k+1)}) = f(ov_{i1})$ if k is even,

$f(v_{ik}v_{i(k+1)}) = f(v_{i1}v_{i2})$ if k is odd,

$f(o) = 5$, $f(v_{i1}) = f(ov_{(i-1)1})$, for $i = 1, 2, 3, 4$, $f(v_{i2}) = f(ov_{i1})$, for $i = 1, 2, 3, 4$,

$f(v_{ik}) = f(v_{i1})$ if k is odd, and $f(v_{ik}) = f(v_{i2})$ if k is even,

$f(v_{21}v_{31}) = f(v_{11}v_{41}) = 6$. Then $\chi_t(S_{4,n}) \leq 6$. ■

Theorem 4 *For $m \geq 5$, $\chi_t(S_{m,n}) = \Delta(G) + 1$.*

Proof. First $\chi_t(S_{m,n}) \geq \Delta(G) + 1$. Let o be the center and let $V(P_i) = \{v_{i1}, ..., v_{in}\}$ for $i = 1, 2, ..., m$. We define a total coloring as follows:

$f(ov_{i1}) = i$ for $i = 1, 2, ..., m$,

$f(v_{i1}v_{(i+1)1}) = f(ov_{i+2})$, for $i = 1, 2, ..., m$, where the addition is calculated in modulo m,

$f(v_{ik}v_{(i+1)k}) = f(v_{i1}v_{(i+1)1})$ for $i = 1, 2, ..., m$, and $k = 2, 3, ..., n$.

$f(v_{i1}v_{i2}) = f(ov_{(i-2)1})$ for $i = 1, 2, ..., m$, $f(v_{i2}v_{i3}) = \Delta(G) + 1$ for $i = 1, 2, ..., m$,

$f(v_{ik}v_{i(k+1)}) = f(v_{i1}v_{i2})$ if k is odd, and $f(v_{ik}v_{i(k+1)}) = f(v_{i2}v_{i3})$ if k is even,

$f(o) = \Delta(G) + 1$,

$f(v_{i1}) = f(ov_{(i-1)1})$ for $i = 1, 2, ..., m$,

$f(v_{i2}) = f(ov_{i1})$, for $i = 1, 2, ..., m$, where the addition is calculated in modulo m,

$f(v_{ik}) = f(vv_{i1})$ if k is odd, and $f(v_{i2})$ if k is even.

So we use $\Delta(G) + 1$ colors, then $\chi_t(S_{m,n}) \leq \Delta(G) + 1$. Then $\chi_t(S_{m,n}) = \Delta(G) + 1$. ■

By Theorem 2 and Theorem 4, we have:

Corollary 5 *For any Spider graph S_m with $m \geq 5$, $\chi'(S_m) = \Delta(G)$ and $\chi_t(S_m) = \Delta(G) + 1$.*

References

[1] C. N. Campos, C. P. de Mello, A result on total coloring of powers of cycles, Discrete Applied Math. 155(2007), 585–597.

[2] G. Fana, L. Sunb, The Erdos Sos conjecture for Spiders, Discrete Math. 307(2007), 3055-3062.

[3] D. B. West, Introduction to graph theory, Prentice-Hall, USA, 2001.

[4] H. P. Yap, Total colorings of graphs in: Lecture Notes in Mathematics, vol. 1623, Springer-Verlage, Berlin, 1996.

Received: September, 2008