

The Exact Controllability of the Movement of Homogeneous Incompressible Viscous Fluid in an Infinite Cylinder

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Abstract

In the present work we propose to study the movement of a steady density of an incompressible viscous fluid in a cylindrical domain to control the external boundary. This movement with axial symmetry, that requests to state that the speed vector is oriented in the tangential direction, this will lead us to transform the system of equations into a linear equation in one dimension. The theory of the moments applied to the deduced linear system, enables us to show the exact controllability of the movement of the incompressible viscous fluid.

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1 Introduction

A lot of problems in mechanics can be solved by using a partial derivative equation which takes into account the various characteristics of the problems (conservation of energy and of momentum...). When one can act on these systems for example while controlling there, speed of movement, we speak about control of the problem, the theory of the control in which we are interested is that which allows, starting of known initial data to reach known final data ones in a time t . If one can find such a control, the system is called then exactly controllable.

In this present work we propose to study the movement of an incompressible viscous fluid in the domain

$$\Omega = \{x \in \mathbf{R}^2 \mid 0 < |x| < R_1\}.$$

This movement subjected to a control $g(t, x)$ on the external boundary:

$$\Gamma_1 = \{x \in \mathbf{R}^2 \mid |x| = R_1\}.$$

For the problems of control we refer the reader to a various works, the results of Fattorini [4] for the heat equation, Coron [1], Fursikov and Imanuvilov [5] for Navier Stokes equation in two dimension, Russell [10], Fursikov and Imanuvilov [6] for the parabolic equations. Different methods are used to solve the problem of exact controllability, that which we will prove in our work. This is the moment theory which has been exposed by Krabs [7]. Other authors use the Hilbert Unicity Method similar to Crépeau [2],[3] for the two equations of Boussinesq and Korteweg-de Vries and Lions for the wave equation, the numerical work are known on the questions For example Zuazua [13].

We consider the system of equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} - \mu \Delta \vec{v} = -\nabla p, (1.1)$$

$$\nabla \cdot \vec{v} = 0. (1.2)$$

In Ω and with the conditions

$$\vec{v} = \vec{g} \quad \text{sur } \Gamma_1, (1.3)$$

$$\vec{v}(0, x) = \vec{v}_0(x). (1.4)$$

The system of equations (1.1) – (1.2) is known like equations of Navier-Stokes. In (1.3) $\vec{g} = \vec{g}(t, x)$ is the control, whose value is a tangential vector (compared to laughed Γ_1).

Further, we will consider the movement with axial symmetry, which means that the vector speed of a movement is in the tangential direction, which allows us to transform the system from equations to an equation in dimension 1. The goal of this article is to show the exact controllability of this movement of the incompressible viscous fluid.

2 Passage to the linear problem of dimension 1

We introduce the polar co-ordinates now

$$x = r \cos \theta, \quad y = r \sin \theta. \quad (2.1)$$

It is pointed out that the change of variables (2.1) implies that

$$\frac{\partial}{\partial x_1} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial x_2} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \quad (2.2)$$

We represent the speed vector $\vec{v}(t, x, y)$ in two components, tangential component u and that radial w

$$\vec{v}(t, x, y) = u(t, x, y)\vec{e}_\theta + w(t, x, y)\vec{e}_r, \quad (2.3)$$

where

$$\vec{e}_\theta = \begin{pmatrix} -y/r \\ x/r \end{pmatrix}, \quad \vec{e}_r = \begin{pmatrix} x/r \\ y/r \end{pmatrix}.$$

In other terms, we express $\vec{v} = (v_1, v_2)$, u and w similar to functions of the polar co-ordinates r and ϑ one has

$$\begin{cases} v_1(t, r, \theta) = -\sin \theta u(t, r, \theta) + \cos \theta w(t, r, \theta), \\ v_2(t, r, \theta) = \cos \theta u(t, r, \theta) + \sin \theta w(t, r, \theta). \end{cases} \quad (2.4)$$

Let us notice that if u and w do not depend on θ , then $w(r, \theta)$ it becomes null and the density does not depend on time. Precisely we have the following assertion.

Remark 2.1 *If \vec{v} satisfies the system of equations (1.1) – (1.2), and if the tangential component $u(t, r, \vartheta)$ and the radial component $w(t, r, \vartheta)$ defined by (2.4) do not depend on θ , then we have*

$$w(t, r, \vartheta) = 0. \quad (2.5)$$

Proof. As by assumption $u(t, r, \vartheta)$ and $w(t, r, \vartheta)$ do not depend on ϑ , then by the relations (2.2), (2.4), we obtain ■

$$\frac{\partial v_1}{\partial x_1} = \cos^2 \vartheta \frac{\partial w}{\partial r} + \frac{1}{r} \sin^2 \vartheta w - \cos \vartheta \sin \vartheta \frac{\partial u}{\partial r} + \frac{1}{r} \cos \vartheta \sin \vartheta u,$$

$$\frac{\partial v_2}{\partial x_2} = \sin^2 \vartheta \frac{\partial w}{\partial r} + \frac{1}{r} \cos^2 \vartheta w + \cos \vartheta \sin \vartheta \frac{\partial u}{\partial r} - \frac{1}{r} \cos \vartheta \sin \vartheta u.$$

We have

$$\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = \frac{\partial w}{\partial r} + \frac{1}{r} w = \frac{1}{r} \frac{\partial}{\partial r}(rw).$$

Consequently, from (1.2) we deduce that

$$\frac{1}{r} \frac{\partial}{\partial r}(rw) = 0,$$

Or

$$rw = C^{te}.$$

However, the $w = 0$ on Γ_1 , it follows that

$$w = 0.$$

Moreover, if u and w do not depend on θ , the problem (1.1), (1.2), and (1.3) is reduced to problem (2.6) – (2.9) as below. More precisely we have:

Theorem 2.1 *Suppose that the tangential component u and that radial w of speed vector do not depend on the θ , then $w = 0$ and u satisfies*

$$u'_t = u''_{r,2} + \frac{1}{r} u'_r - \frac{1}{r^2} u, \quad Q = [0, R_1] \times [0, T] \quad (2.6)$$

$$u(0, r) = u_0(r), \quad (2.7)$$

$$u(0, t) = 0, \quad (2.8)$$

$$u(R_1, t) = \omega(t). \quad (2.9)$$

Proof. As we observed in the remark before, if the tangential component u and the radial w of the speed vector do not depend on ϑ , then we have $w = 0$.

So

$$\vec{v}(t, r) = u(t, r) \begin{pmatrix} -y/r \\ x/r \end{pmatrix},$$

where

$$v_1(t, x) = u(t, r) \frac{-y}{r}, \quad v_2(t, x) = u(t, r) \frac{x}{r}.$$

We have

$$\begin{aligned} \Delta v_1(x, y) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(t, r) \frac{-y}{r} = \\ &= -\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial r} \frac{\partial r}{\partial x} \frac{y}{r} - u \frac{y}{r^2} \frac{\partial r}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial r} \frac{\partial r}{\partial y} \frac{y}{r} + \frac{u}{r} - u \frac{y}{r^2} \frac{\partial r}{\partial y} \right) \\ &= -\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial r} \frac{xy}{r^2} - u \frac{xy}{r^3} \right) - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial r} \frac{y^2}{r^2} + \frac{u}{r} - u \frac{y^2}{r^3} \right) \\ &= -\frac{\partial^2 u}{\partial r^2} \frac{x^2 y}{r^3} - \frac{\partial u}{\partial r} \frac{y}{r^2} + 2 \frac{\partial u}{\partial r} \frac{x^2 y}{r^4} + \frac{\partial u}{\partial r} \frac{x^2 y}{r^4} + u \frac{y}{r^3} - 3u \frac{x^2 y}{r^5} \\ &\quad - \frac{\partial^2 u}{\partial r^2} \frac{y^3}{r^3} - 2 \frac{\partial u}{\partial r} \frac{y}{r^2} + 2 \frac{\partial u}{\partial r} \frac{y^3}{r^4} - \frac{\partial u}{\partial r} \frac{1}{r} + u \frac{y}{r^3} + \frac{\partial u}{\partial r} \frac{y^3}{r^4} + 2u \frac{y}{r^3} - 3u \frac{y^3}{r^5} \\ &= -\left(\frac{\partial^2 u}{\partial r^2} \frac{y}{r} + \frac{\partial u}{\partial r} \frac{y}{r^2} - u \frac{y}{r^3} \right) = -\frac{y}{r} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} u \right) = -\frac{y}{r} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r u}{\partial r} \right) \right). \end{aligned}$$

■

For $\Delta v_2(x, y)$, thanks to the isotropy of the operator of Laplace Δ , we have obviously

$$\Delta v_2(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(t, r) \frac{x}{r} = \frac{x}{r} \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r u}{\partial r} \right) \right).$$

(Naturally we can deduce this relationship by explicitly calculating each term as for calculation from $\Delta v_2(x, y)$.

And by calculates explicit and by taking account the relations $\cos \vartheta = \frac{x}{r}$, $\sin \vartheta = \frac{y}{r}$, the nonlinear $(\vec{v} \cdot \nabla)\vec{v}$ term is

$$(\vec{v} \cdot \nabla)\vec{v} = -u(t, r)\sin \vartheta \frac{\partial u}{\partial r} \frac{x}{r} + u(t, r)\cos \vartheta \frac{\partial u}{\partial r} \frac{y}{r} = 0. (2.10)$$

This is completed the demonstration.

3 Functional framework

For our problem it is significant to introduce two Hilbert spaces $L_r^2(0, R_1)$ and H . We define initially the Hilbert space

$$L_r^2(0, R_1) = \{ u \text{ mesurable} \mid \int_0^{R_1} r|u(r)|^2 dr < \infty \}, (3.1)$$

Provided with the scalar product

$$\langle u, v \rangle_{L_r^2(0, R_1)} = \int_0^{R_1} ru(r)v(r)dr. (3.2)$$

We define moreover the Hilbert space

$$H = \{ u \text{ mesurable} \mid u(0) = u(R_1) = 0 \text{ et } \int_0^{R_1} \frac{1}{r} \left| \frac{\partial ru}{\partial r} \right|^2 < \infty \}, (3.3)$$

with the scalar product

$$\langle u, v \rangle_H = \int_0^{R_1} \frac{1}{r} \frac{\partial ru(r)}{\partial r} \frac{\partial rv(r)}{\partial r} dr. (3.4)$$

It is noticed that H is contained in $L_r^2(0, R_1)$. Indeed for $u \in H$ we have

$$\int_0^{R_1} r|u(r)|^2 dr = \int_0^{R_1} \left(\frac{1}{\sqrt{r}} ru(r) \right)^2 dr = \int_0^{R_1} \left(\frac{1}{\sqrt{r}} \int_0^r \frac{\partial r' u}{\partial r'} dr' \right)^2 dr \leq$$

$$\leq \int_0^{R_1} \left(\int_0^r \frac{1}{\sqrt{r'}} \frac{\partial r' u}{\partial r'} dr' \right)^2 dr \leq R_1 \|u\|_H^2.$$

Proposition 3.1 *The injection of H in $L_r^2(0, R_1)$ is compact.*

Proof. Obviously the parts limited in H remain limited in $L_r^2(0, R_1)$ and to be able to apply the theorem of Arzela we should show their equicontinuity. For that, we consider a part E limited of H and $u \in E$. ■
Let us consider the difference

$$\begin{aligned} \sqrt{r_1}u(r_1) - \sqrt{r_0}u(r_0) &= \int_{r_0}^{r_1} \frac{\partial \sqrt{r}u(r)}{\partial r} dr = (3.5) \\ &= \int_{r_0}^{r_1} \left(\frac{1}{\sqrt{r}} \frac{\partial ru(r)}{\partial r} - \frac{1}{2} \frac{1}{\sqrt{r}} u(r) \right) dr. \end{aligned}$$

It is obvious that

$$\begin{aligned} \left| \int_{r_0}^{r_1} \frac{1}{\sqrt{r}} \frac{\partial ru(r)}{\partial r} dr \right| &\leq \left(\int_{r_0}^{r_1} 1 dr \right)^{\frac{1}{2}} \left(\int_{r_0}^{r_1} \frac{1}{\sqrt{r}} \left(\frac{\partial ru(r)}{\partial r} \right)^2 dr \right)^{\frac{1}{2}} \leq (3.6) \\ &\leq (r_1 - r_0)^{\frac{1}{2}} \|u\|_H. \end{aligned}$$

Let us consider now

$$\int_{r_0}^{r_1} \left(\frac{1}{\sqrt{r}} u(r) \right) dr.$$

Under the terms of the inequality of Holder we have

$$\left| \int_{r_0}^{r_1} \frac{1}{\sqrt{r}} u(r) dr \right| \leq (r_1 - r_0)^{\frac{1}{4}} \left(\int_{r_0}^{r_1} \left| \left(\frac{1}{\sqrt{r}} u(r) \right) \right|^{\frac{4}{3}} dr \right)^{\frac{3}{4}}, (3.7)$$

Like

$$\frac{1}{\sqrt{r}} u(r) = \frac{1}{\sqrt{r^3}} ru(r) = \frac{1}{\sqrt{r^3}} \left(\int_0^r \frac{\partial r' u'(r)}{\partial r'} dr' \right).$$

We have

$$\begin{aligned}
\left| \frac{1}{\sqrt{r}} u(r) \right| &= \left| \frac{1}{\sqrt{r^3}} \int_0^r \frac{\partial r' u'(r)}{\partial r'} dr' \right| \leq (3.8) \\
&\leq \frac{1}{r} \left| \int_0^r \frac{1}{\sqrt{r'}} \frac{\partial r' u'(r)}{\partial r'} dr' \right| \leq \frac{1}{r} \left(\int_0^r 1 dr' \right)^{1/2} \left(\int_0^r \frac{1}{r'} \left(\frac{\partial r' u'(r)}{\partial r'} \right)^2 dr' \right)^{1/2} \leq \\
&\leq \frac{1}{r} r^{1/2} \|u\|_H = \frac{1}{\sqrt{r}} \|u\|_H.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{r_0}^{r_1} \left| \frac{1}{\sqrt{r}} u(r) \right|^{4/3} dr &\leq \int_{r_0}^{r_1} \left| \frac{1}{\sqrt{r}} \|u(r)\|_H \right|^{4/3} dr \leq (3.9) \\
&\leq \|u\|_H \left(\int_{r_0}^{r_1} \frac{1}{\sqrt{r^3}} dr \right)^{3/4} \leq \|u\|_H \left(\int_0^{R_1} \frac{1}{\sqrt{r^3}} dr \right)^{3/4} = \|u\|_H \left(\frac{r^{1/3}}{1/3} \Big|_0^{R_1} \right)^{3/4} = 3^{3/4} R_1^{1/4} \|u\|_H.
\end{aligned}$$

Enjoining (3.5), (3.6), (3.7), (3.8) and (3.9) we obtain

$$\left| \sqrt{r_1} u(r_1) - \sqrt{r_0} u(r_0) \right| \leq (r_1 - r_0)^{1/2} \|u\|_H + (r_1 - r_0)^{1/4} 3^{3/4} R_1^{1/4} \|u\|_H (3.10).$$

Set

$$K = \sup_{u \in E} \|u\|_H,$$

We have obviously

$$\|u\|_H \leq K < \infty \quad \forall u \in E.$$

Thus, we recalling arbitrarily of r_0, r_1 , in $[0, R_1]$, we see that the relationship (3.10) implies that E is equicontinuous.

To solve our problem of control, we consider the properties of the operator A which defines (- the equation (2.6). This operator can be written

$$A(\cdot) = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \cdot) \right).$$

And as it is not limited, one will be interested in the properties of its operator inverse in order to establish the existence of a base hilbertienne in our basic space $L_r^2(0, R_1)$.

Let us consider the equation

$$-\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r u}{\partial r} \right) = f, \quad (3.11)$$

where f is an element of $L_r^2(O, R_1)$, with the boundary conditions.

$$u(0) = u(R_1) = 0. \quad (3.12)$$

In the next will show that the inverse operator A exists and it is self-adjoint and compact in space $L_r^2(O, R_1)$.

Proposition 3.2 *For all $f \in L_r^2(O, R_1)$, the problem (3.11) – (3.12) admits a weak solution and only one in H , i.e., there exists an element u and only one in H that satisfies*

$$\langle u, v \rangle_H = \langle f, v \rangle_{L_r^2(0, R_1)} \quad \forall v \in H. \quad (3.13)$$

Proof. Like we noticed in top, we have $\|v\|_{L_r^2(0, R_1)} \leq C \|v\|_H$. Thus the scalar product $\langle f, v \rangle_{L_r^2(0, R_1)}$ defines a continuous linear functional in H (for $v \in H$). That being, as there is well-known, according to the theorem of Riesz, it exist an element u in H and only one verifying (3.13). ■

The Proposition (3.2) we make us to define the operator which, $f \in L_r^2(0, R_1)$, associates the element $u \in H$, H verifying (3.11). As this element $u \in H$ is the weak solution of the problem (3.12) – (3.13), let us indicate it by A^{-1} .

Proposition 3.3 *The operator A^{-1} is an operator auto-adjoint and compact in space $L_r^2(0, R_1)$.*

Proof. As A^{-1} can be considered as the inverse operator of A and that operator A with the boundary conditions (3.13) is an operator self adjoint in its field of definition, it is seen easily that A^{-1} is also self adjoint an operator . ■

Let us check now that A^{-1} is a compact operator. Indeed we have, appreciation to (3.13). What can be written.

$$\|A^{-1}f\|_H^2 = \langle u, u \rangle_H = \langle f, u \rangle_{L_r^2(0, R_1)} \leq \|f\|_{L_r^2(0, R_1)} \|u\|_{L_r^2(0, R_1)}.$$

What can be written

$$\|A^{-1}f\|_H^2 \leq C \|f\|_{L_r^2(0, R_1)}.$$

And as the injection of H in $L_r^2(0, R_1)$ is compact, we deduce that A^{-1} is a compact operator. In addition there is the kernel of A^{-1} reduce to the null vector since

$$A^{-1}f = 0, \text{ then } f = 0.$$

In order to approach the qualitative study of control we state a theorem existence and of unicity of the traditional solution as follows.

Theorem 3.1 *For all initial condition*

$$u_0 \in L_r^2(0, R_1) \quad (3.14)$$

the linear problem (2.6), (2.7), (2.8) and (2.9) admit a unique solution

$$u(\cdot, t) \in L_r^2([0, R_1]) \quad \forall t \in [0, T],$$

for each r

$$r \in [0, R_1], \quad u(r, \cdot) \in L_r^2(0, R_1),$$

and

$$\|u(r, 0) - u_0(r)\|_{L_r^2(0, R_1)} = 0.$$

4 Exact controllability and problem of moment

In this paragraph we study the controllability of the system. More precisely we will show the null controllability, i.e., when the final state reached is $u_T(r) = 0$.

Definition 4.1 *It is said that the problem (2.6), (2.7), (2.8) and (2.9) is null exactly controllable in time T , if being given conditions initial and final $u_0(r)$ and $u_T(r) = 0$ in $L_r^2(0, R_1)$, we can find a control $\omega(t) \in L^2([0, T])$, which makes pass the system of $u_0(r)$ in a final state 0.*

4.1 Reduction to moment problems

While using the properties established for the operator $-A$ in $L_r^2(0, R_1)$ in section 2, space $L_r^2(0, R_1)$ will have a base of eigenfunction associated to eigenvalues of the operator $(-A)^{-1}$. The following lemmas gets a method of calculation of these eigenvalues and vectors eigenfunction associated.

Lemma 4.1 *Eigenvalues of the operator $-A$ are them $(-\lambda_n)$ equal to $(\frac{1}{\mu_n})$ with (μ_n) eigenvalues of operator $(-A)^{-1}$ the eigenfunctions $\varphi_n(r)$ are the solutions of the Bessel equation*

$$(\varphi_n)''(r) + \frac{1}{r}(\varphi_n)'(r) + \left(-\frac{1}{r^2} + \lambda_n\right)\varphi_n(r) = 0. \quad (4.1)$$

Proof. The demonstration is obvious being given that the operator $(-A)^{-1}$ is self-adjoint and compact in $L_r^2(0, R_1)$ it admits nonnegative eigenvalues thus.

■
Set

$$-A\varphi_n(r) = -\lambda_n\varphi_n(r),$$

this is equivalent to (4.1). Set

$$\varrho = \sqrt{\lambda_n}r, \Psi_n(\varrho) = \varphi_n(r), \varphi_n'(r) = \Psi_n'(\varrho)\sqrt{\lambda_n}, \varphi_n''(r) = \Psi_n''(\varrho)\lambda_n,$$

we have

$$\Psi_n''(\varrho) + \frac{1}{\varrho}\Psi_n'(\varrho) + \left(1 - \frac{1}{\varrho^2}\right)\Psi_n(\varrho) = 0. \quad (4.2)$$

Thus, we obtain, the Bessel equation of index 1 ($p = 1$) and the solutions of (4.2) are functions to Bessel of index 1 with constant near

$$\varphi_n(r) = J_1(\sqrt{\lambda_n}r)$$

$$= J_1(\sqrt{\lambda_n}r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+k)!} \left(\frac{\sqrt{\lambda_n}r}{2}\right)^{1+2k}.$$

Lemma 4.2 [11] *The functions $\varphi_n(r)$ form an orthonormal basis for Hilbert space $L_r^2(0, R_1)$, in order hand it is well known, that the functions $J_1(\varrho_n)$ admit infinity of zeros i.e. the sequence (ϱ_n) such as $\varrho_n < \varrho_n + 1, n = 1, 2, \dots$ and*

$$J_1(\varrho_n) = 0, \quad (4.3)$$

then, the eigenvalues are roots of the equation

$$J_1(\sqrt{\lambda_n} R_1) = 0. \quad (4.4)$$

And all considers of it the following expression of the Bessel function of index 1

$$J_1(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{3\pi}{4}\right), \quad (4.5)$$

the check that

$$\lambda_n \approx \frac{n^2 \pi^2}{R_1^2} \quad n \in \mathbf{N}. \quad (4.6)$$

In the final step we show the exact controllability applying the moment method.

Theorem 4.1 Suppose that $T > 0$, and

$$u_0(r) = \sum_{n=1}^{\infty} \mu_n \varphi_n(r), \quad \sum_{n=1}^{\infty} \mu_n^2 < \infty.$$

Then it exist a control of $\omega(t) \in L^2(0, T)$ such solution $u(r, t)$ of (2.6)–(2.7)–(2.9) satisfy $u(r, T) = 0$ if and only if it exist a function $\omega_1(t)$ which verify the following

$$\int_0^T \omega_1(t) \exp(-\lambda_n t) dt = \frac{\mu_n \exp(-\lambda_n T)}{R_1 \varphi_n'(R_1)}, \quad \forall n \in N \setminus \{0\}. \quad (4.7)$$

Proof. For showing the theorem, we considers the adjoint problem ■

$$v_t' + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r v}{\partial r} \right) = 0 \quad \text{pour } r \in [0, R_1], \quad t \in [0, T] \quad (4.8)$$

$$v(0, t) = 0, v(R_1, t) = 0 \quad (4.9)$$

$$u(r, t) = \sum_{n=1}^{\infty} \eta(t) \varphi_n(r) \quad \sum_{n=0}^{\infty} \eta_n^2(t) < \infty \quad (4.10)$$

The solution of the problem (4.8) – (4.9) is written

$$v(r, t) = \varphi_n(r) \exp(\lambda_n(t - T)).$$

Let us calculate

$$\int_0^T \int_0^{R_1} rv \left(u'_t - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial ru}{\partial r} \right) \right) dr dt = 0 \quad (4.11)$$

set

$$I_1 = \int_0^T \int_0^{R_1} rv u'_t dr dt$$

$$I_2 = \int_0^T \int_0^{R_1} rv \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial ru}{\partial r} \right) dr dt$$

$$I_1 = \int_0^{R_1} \int_0^T rv u'_t dt dr = \int_0^{R_1} ([ruv]_0^T - \int_0^T ruv'_t dt) dr$$

$$= \int_0^{R_1} [rv(r, T)u(r, T) - rv(r, 0)u(r, 0)] dr - \int_0^{R_1} \int_0^T ruv'_t dr dt$$

$$= - \int_0^{R_1} r \varphi_n(r) \exp(-\lambda_n T) u_0(r) dr - \int_0^{R_1} \int_0^T ruv'_t dr dt$$

$$I_2 = \int_0^T ([rv \frac{1}{r} \frac{\partial ru}{\partial r}]_0^{R_1} - \int_0^{R_1} \frac{1}{r} \frac{\partial ru}{\partial r} \frac{\partial rv}{\partial r} dr) dt$$

$$= \int_0^T (v(R_1, t) \frac{\partial ru}{\partial r}(R_1, t) - v(0, t) \frac{\partial ru}{\partial r}(0, t) - \int_0^{R_1} \frac{1}{r} \frac{\partial ru}{\partial r} \frac{\partial rv}{\partial r} dr) dt.$$

Then (4.11) is equivalent to

$$I_1 - I_2 = - \int_0^{R_1} r\varphi_n(r) \exp(-\lambda_n T) u_0(r) dr - \int_0^T \int_0^{R_1} \left(ruv'_t + ru \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial rv}{\partial r} \right) - ru \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial rv}{\partial r} \right) - \frac{1}{r} \frac{\partial ru}{\partial r} \frac{\partial rv}{\partial r} \right) dr dt = 0.$$

It follows that

$$\int_0^{R_1} r\varphi_n(r) \exp(-\lambda_n T) u_0(r) dr = \int_0^T \int_0^{R_1} \left(ru \frac{1}{r} \frac{\partial rv}{\partial r} + \frac{1}{r} \frac{\partial ru}{\partial r} \frac{\partial rv}{\partial r} \right) dr dt,$$

and

$$\begin{aligned} \int_0^{R_1} r\varphi_n(r) \exp(-\lambda_n T) u_0(r) dr &= \int_0^T \int_0^{R_1} \frac{\partial}{\partial r} \left(ru \frac{1}{r} \frac{\partial ru}{\partial r} \right) dr dt \\ &= \int_0^T \left[\frac{1}{r} \frac{\partial ru}{\partial r} ru \right]_0^{R_1} dt, \end{aligned}$$

and

$$\begin{aligned} \int_0^{R_1} r\varphi_n(r) \exp(-\lambda_n T) u_0(r) dr &= \int_0^T \omega(t) (R_1 v'_r(R_1, t) + v(R_1, t)) dt = \\ &= \int_0^T \omega(t) R_1 \varphi'_n(R_1, t) \exp(-\lambda_n(t - T)) dt. \end{aligned}$$

By putting $\omega_1(t) = \omega(T - t)$, we will obtain the equation(4.7).

Remark 4.1 The relation (4.7) is called the moment equation for our problem.

4.2 Resolution of moment problems

Let us consider the problem of moment given higher (4.7) in the Hilbert space $L^2(0, T)$, to solve it we present the theory of Muntz concerning the exponential sums.

Theorem 4.2 [12] *Let us suppose the condition*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < +\infty. (4.12)$$

Then the family $(\exp(-\lambda_n t))_{n=1,2,\dots}$ is free. Moreover, we can build a biorthogonal sequence $\theta_n(t)$, $n = 1, 2, \dots$ of family Λ in the space $L^2(0, T)$, which verify the following condition

$$\int_0^T \theta_n(t) \exp(-\lambda_n t) dt = \sigma_{nm}, \tag{4.13}$$

where σ_{nm} is the symbol of Kroneker.

Proposition 4.1 *The condition (4.12) is checked for the problem (2.6) – (2.9). It exist then a biorthogonal sequence to family $(\theta_n(t))_{n=1,2,\dots}$ and a function $\omega_1(t)$ is the solution of moment equation (4.7) given by,*

$$\omega_1(t) = \sum_{n=1}^{\infty} \frac{\mu_n \exp(-\lambda_n T) \theta_n(t)}{R_1 \varphi'_n(R_1)} \text{ for } n \geq 1 \tag{4.14}$$

with

$$\varphi'_n(R_1) = J'_1(\sqrt{\lambda_n} R_1).$$

Proof. The eigenvalues checked the condition (4.12) and according to the Muntz theory given in [4] and [12] we can set, $A(\Lambda, T)$ Adherent of underneath space generated by the family $\exp(-\lambda_n t)$ and $A(m, \Lambda, T)$ Adherent of underneath space generated by the family $(\exp(-\lambda_n t))_{n \neq m}$

$$p_m(t) = \exp(-\lambda_m t), \quad p_m(t) \notin A(m, \Lambda, T)$$

r_m is the projection of $p_m(t)$ on $A(m, \Lambda, T)$. ■

Our biorthogonal sequence can be chosen as follows:

$$\theta_m(t) = \frac{\exp(-\lambda_m t) - r_m}{\|\exp(-\lambda_m t) - r_m\|_{L^2(0,T)}^2}. \tag{4.15}$$

We substitute (4.15) in moment equation so we establish the formula (4.14).

Remark 4.2 *The formula (4.14) is the formal expression of control $\omega_1(t)$. We provide the theory of exact controllability.*

Theorem 4.3 *Let $T > 0$ and the initial condition,*

$$u_0(r) = \sum_{n=1}^{\infty} \mu_n \varphi_n(r) \quad \sum_{n=0}^{\infty} \mu_n^2 < \infty$$

It exists a control $\omega_1(t) \in L^2(0, T)$ such as the solution $u(r, t)$ from the problem(2.6), (2.7), (2, 8) and (2.9) satisfied with $u(r, T) = 0$.

Proof. We consider the formal solution (4.14) we will show that it belongs well to space $L^2(0, T)$, for that a set ■

$$\omega_1(t) = \sum_{n=1}^{\infty} \langle \omega_1(t), \exp(-\lambda_n t) \rangle \theta_n(t).$$

By analogy with what was made for the heat equation in [4] and by considering the estimate of the eigenvalues $(\lambda_n)_n$ by $\frac{n^2\pi^2}{R_1^2}$ we can verified that,

$$\|\theta_n(t)\|_{L^2(0,\infty)} \leq \frac{\sqrt{2n\pi} M e^{\varpi n}}{R_1}$$

M and ϖ two positive constants. Moreover, by applying the invertible property of the restriction operator,

$$R_T : E(\Lambda, \infty) \rightarrow E(\Lambda, T), R_T(v) = v_{[0,T]}.$$

established if and only if

$$\|R_T^{-1}\| \leq C.$$

We have

$$\|\theta_n(t)\|_{L^2(0,T)} \leq M' e^{\varpi n},$$

where $M' = M'(T)$ then the series (4.14) converge absolutely and

$$\begin{aligned} \|\omega_1(t)\|_{L^2(0,T)} &\leq \sum_{n=1}^{\infty} \left\| \frac{\mu_n \exp(-\lambda_n T) \theta_n(t)}{R_1 \varphi'_n(R_1)} \right\|_{L^2(0,T)} \leq \sum_{n=1}^{\infty} \frac{|\mu_n| \exp(-\lambda_n T) M' e^{\varpi n}}{R_1 |\varphi'_n(R_1)|} \leq \\ &\leq \frac{M'}{R_1} \sum_{n=1}^{\infty} \frac{|\mu_n| e^{(\varpi n - \lambda_n T)}}{|\varphi'_n(R_1)|} < \infty. \end{aligned}$$

with $(\varphi_n)_n$ eigenfunctions of operator A associated with the eigenvalues which are in $L^2_r(0, R_1)$ space. Thus our control $\omega_1(t)$ is well in $L^2(0, T)$. What completes the demonstration.

Conclusion 1 *In conclusion, in this paper once more we have confirmed the results of Fursikov in [4] for the exact controllability of the Naviers-Stokes system in the case of a disc with a radius R_1 applying the theory of the moments to the system transformed into polar coordinates, the initial condition supposed in this work is such as its transform in the Hilbert space $L_r^2(0, R_1)$. We hope to be able to show the exact controllability of the system of Naviers-Stokes with a variable density using the same technique.*

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