

Application of Variational Iteration Method to Parabolic Problems

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Abstract

In this paper, the solutions of the one dimensional non-homogeneous parabolic equations with a variable coefficient are obtained by means of variational iteration method (VIM). The results reveal that the variational iteration method (VIM) is very effective, convenient and quite accurate to systems of partial differential equations. It is predicted that the VIM can be found widely applicable in engineering.

Keywords: partial differential equation; one dimensional non-homogeneous parabolic equations; variational iteration method.

1 Introduction

Most models of real – life problems are still very difficult to solve. Therefore, approximate analytical solutions such as variational iteration method [1-8] were introduced.

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This method is the most effective and convenient ones for both linear and nonlinear equations.

The VIM is to construct correction functional using general Lagrange multipliers identified optimally via the the variational theory, and the initial approximation can be freely chosen with unknown constants.

In this paper, we apply variational iteration method (VIM) to one-dimensional non-homogeneous parabolic partial differential equations with a variable coefficient of the form [9]:

$$\frac{\partial u}{\partial t} = \mu(x) \frac{\partial^2 u}{\partial x^2} + \phi(x, t), \quad 0 < x < 1, t > 0, \quad (1)$$

With initial condition:

$$u(x, 0) = f(x), \quad (2)$$

and boundary conditions:

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t). \quad (3)$$

2 Variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \quad (4)$$

Where L is a linear operator, N is a nonlinear operator and $g(t)$ is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda (Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)) d\tau \quad (5)$$

Where λ is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the n th approximation and u_n is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$

3 Example 1

Let us consider the problem [9]:

$$u_t = u_{xx} + \phi(x, t) = u_{xx} + \exp(-x)(\cos t - \sin t), \quad (6)$$

where the initial condition is $u(x, 0) = f(x) = x$ and the boundary conditions are

$$u(0, t) = \sin t, \quad u(1, t) = 1 + \frac{\sin t}{e} \tag{7}$$

Which is easily seen to have the exact solution $u(x, t) = x + \exp(-x) \sin t$ [9].

3.1 Application of variational iteration method

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left(\left(\frac{\partial u_n(x, \tau)}{\partial \tau} \right) - \left(\frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right) - e^{-x} \cos \tau + e^{-x} \sin \tau \right) d\tau \tag{8}$$

We obtain the lagrangian multiplier:

$$\lambda = -1 \tag{9}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t (-1) \left(\left(\frac{\partial u_n(x, \tau)}{\partial \tau} \right) - \left(\frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right) - e^{-x} \cos \tau + e^{-x} \sin \tau \right) d\tau \tag{10}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x, t) = x, \tag{11}$$

Using the above variational formula (10), we have

$$u_1(x, t) = u_0(x, t) + \int_0^t (-1) \left(\left(\frac{\partial u_0(x, \tau)}{\partial \tau} \right) - \left(\frac{\partial^2 u_0(x, \tau)}{\partial x^2} \right) - e^{-x} \cos \tau + e^{-x} \sin \tau \right) d\tau \tag{12}$$

Substituting Eq. (11) in to Eq. (12) and after simplifications, we have:

$$u_1(x, t) = x - e^{-x} + e^{-x} \sin t + e^{-x} \cos t \tag{13}$$

In the same way, we obtain $u_2(x, t), u_3(x, t), u_4(x, t), u_5(x, t)$ as follows:

$$u_2(x, t) = x + 2e^{-x} \sin t - e^{-x} t \tag{14}$$

$$u_3(x, t) = x + e^{-x} \sin t + e^{-x} - e^{-x} \cos t - \frac{1}{2} e^{-x} t^2 \tag{15}$$

$$u_4(x, t) = x + e^{-x} t - \frac{1}{6} e^{-x} t^3 \tag{16}$$

(17)

$$u_5(x, t) = x - e^{-x} + e^{-x} \sin t + e^{-x} \cos t + \frac{1}{2}e^{-x}t^2 - \frac{1}{24}e^{-x}t^4$$

And so on. In the same way the rest of the components of the iteration formula can be obtained.

Tables 1 and 2 and figs.1 and 2 show that variational iteration method is more efficient than the ADM.

Table 1: Numerical solutions for $x = 0.1$

t	$u(x, t)$ Exact solution	$u(x, t)$ Adomian method	$u(x, t)$ variational iteration method
0.1	0.190333	0.190329	0.190333
0.3	0.367397	0.367093	0.367396
0.5	0.533802	0.531465	0.533782
0.7	0.682912	0.674006	0.682765
0.9	0.808783	0.784705	0.808125

Table 2: Numerical solutions for $x = 0.9$

t	$u(x, t)$ Exact solution	$u(x, t)$ Adomian method	$u(x, t)$ variational iteration method
0.1	0.940589	0.940587	0.940589
0.3	1.020149	1.020012	1.020149
0.5	1.094919	1.093869	1.094911
0.7	1.161919	1.157917	1.161853
0.9	1.218476	1.207658	1.218181

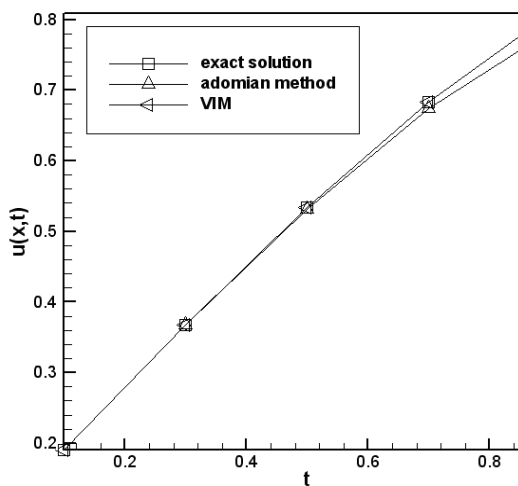


Fig.1.comparison between results of the VIM with Adomian method and exact solution at $x = 0.1$

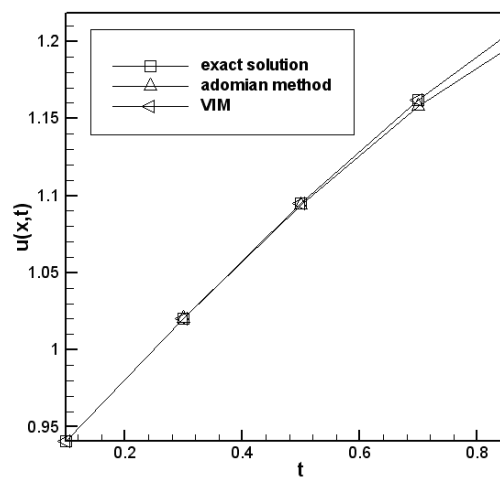


Fig.2.comparison between results of the VIM with Adomian method and exact solution at $x = 0.9$

4 Example 2

Let us consider the problem [9].

$$u_{tt} = u_{xx} + \exp(x)(\cosh t - \sinh t), \tag{18}$$

With the initial and boundary conditions posed are:

$$u(x, 0) = f(x) = \frac{x^3}{6} \tag{19}$$

$$u(0, t) = \sinh t, \quad u(1, t) = e \cdot \sinh t + t + \frac{1}{6},$$

Exact solution of this equation is:

$$u(x, t) = \exp(x) \sinh t + \frac{x^3}{6} + xt. \tag{20}$$

4.1 Application of variational iteration method

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left(\left(\frac{\partial u_n(x, \tau)}{\partial \tau} \right) - \left(\frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right) - e^{-x} \cosh \tau + e^{-x} \sinh \tau \right) d\tau \tag{21}$$

We obtain the lagrangian multiplier:

$$\lambda = -1 \tag{22}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t (-1) \left(\left(\frac{\partial u_n(x, \tau)}{\partial \tau} \right) - \left(\frac{\partial^2 u_n(x, \tau)}{\partial x^2} \right) - e^{-x} \cosh \tau + e^{-x} \sinh \tau \right) d\tau \tag{23}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x, t) = \frac{1}{6}x^3, \tag{24}$$

Using the above variational formula (23), we have

$$u_1(x, t) = u_0(x, t) + \int_0^t (-1) \left(\left(\frac{\partial u_0(x, \tau)}{\partial \tau} \right) - \left(\frac{\partial^2 u_0(x, \tau)}{\partial x^2} \right) - e^{-x} \cosh \tau + e^{-x} \sinh \tau \right) d\tau \tag{25}$$

Substituting Eq. (24) in to Eq. (25) and after simplifications, we have

$$u_1(x, t) = \frac{1}{6}x^3 + e^x + e^x \sinh t - e^x \cosh t + xt \quad (26)$$

In the same way, we obtain $u_2(x, t), u_3(x, t), u_4(x, t), u_5(x, t)$ as follows:

$$u_2(x, t) = \frac{1}{6}x^3 + xt + e^x t \quad (27)$$

$$u_3(x, t) = \frac{1}{6}x^3 + xt + e^x + e^x \sinh t - e^x \cosh t + \frac{1}{2}e^x t^2 \quad (28)$$

$$u_4(x, t) = \frac{1}{6}x^3 + xt + e^x t + \frac{1}{6}e^x t^3 \quad (29)$$

$$u_5(x, t) = \frac{1}{6}x^3 + xt + e^x + e^x \sinh t - e^x \cosh t + \frac{1}{2}e^x t^2 + \frac{1}{24}e^x t^4 \quad (30)$$

And so on. In the same way the rest of the components of the iteration formula can be obtained.

Tables 3 and 4 and figs.3 and 4 show that variational iteration method is more efficient than the ADM

Table 3: Numerical solutions for $x = 0.1$

t	$u(x, t)$ Exact solution	$u(x, t)$ adomian method	$u(x, t)$ variational iteration method
0.1	0.120868	0.120863	0.120868
0.3	0.366713	0.366339	0.366712
0.5	0.626066	0.623163	0.626041
0.7	0.908531	0.897292	0.908349
0.9	1.224649	1.193602	1.223815

Table 4: Numerical solutions for $x = 0.9$

t	$u(x, t)$ Exact solution	$u(x, t)$ Adomian method	$u(x, t)$ variational iteration method
0.1	0.457870	0.457860	0.457870
0.3	1.140499	1.139666	1.140496
0.5	1.853187	1.846728	1.853134
0.7	2.617314	2.592303	2.616909
0.9	3.456323	3.387242	3.454481

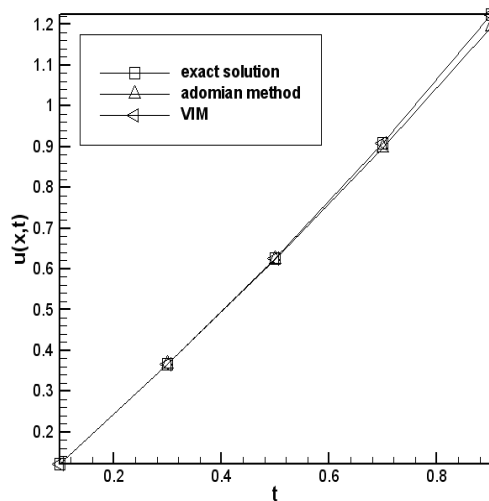


Fig.3.comparison between results of the VIM with adomian method and exact solution at $x = 0.1$

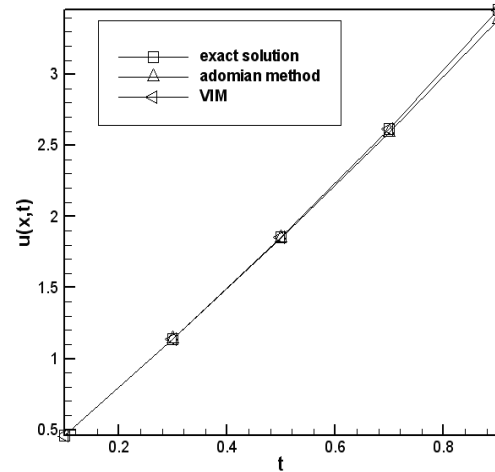


Fig.4.comparison between results of the VIM with adomian method and exact solution at $x = 0.9$

5 Conclusion

In this work, we proposed variational iteration method for solving one dimensional non-homogeneous parabolic equations with a variable coefficient. The results obtained here were compared with results of exact solution and homogeneous Adomian decomposition method. The results revealed that the variational iteration method is more efficient than the ADM and it is a powerful mathematical tool for solutions of differential equations in terms of accuracy and efficiency.

References

- [1] J.H. He, Variational iteration method: a kind of nonlinear analytical technique: some examples, *International Journal of Nonlinear Mechanics*, 344 (1999), 699–708.
- [2] J.H. He, Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation*, 114 (2000), 115–123.
- [3] S. Momani, S. Abusad, Application of He's variational iteration method to Helmholtz equation, *Chaos, Soliton & Fractals*, 27 (2006), 1119–1123.
- [4] A.Barari, Abdoul R.Ghotbi, F.Farrokhzad and D.D.Ganji; Variational iteration method and Homotopy-perturbation method for solving different types of wave equations, *Journal of Applied Sciences*, 8(2008), 120-126.

- [5] A.Barari, M.Omidvar, Abdoul R.Ghotbi and D.D.Ganji; Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations, *Acta Applicandae Mathematicae*, 104(2008), 161-171.
- [6] Z.M.Odibat, S.Momani, Application of variational iteration method to nonlinear differential equations of fractional order, *International Journal of Nonlinear Sciences and Numerical Simulation*, 7(2006), 27–34.
- [7] J.H.He, Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Phys. B*, 20 (2006), 1141–1199.
- [8] J.H.He, X.H.Wu, Construction of solitary solution and compacton-like solution by variational iteration method, *Chaos, Solitons & Fractals*, 29 (2006) 108–113.
- [9] B.Jang, Exact solutions to one dimensional non-homogeneous parabolic problems by the homogeneous adomian decomposition method, *Applied Mathematics and Computation*, 186(2007), 969-979.

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