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# Operational Calculus Method for Some Mixed Boundary Problems

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**Abstract**

Parper devoted to dual integral equations involving nonstatianary heat condition equation in the Laplace transform image (L-transform) for two dimentional symmetrical under mixed discontinious boundary conditions acted on level surface of semi-space in cylindrical coordinates. We consider a nonstationary heat conduction equation to determine a temperature distribution function of moving solid object a long a surface of semi-space with velocity $v$ and heat source $m_1(r,\tau)$ inside a disk of radius $b$, $r<b$, outside the disk $r>b$ a temperature fuction $m_2(r,\tau)$ is given. The solution of the given boundary value problem is given with the aid of operational calculus method and dual integral equations

**Keywords:** Dual integral equation, mixed boudary conditions, heat conduction equation.

## 1. Introduction

Dual integral equations method arises in a study of mixed boundary value problems in mathematical physics equation of elliptic type in different areas of applications such as: potential theory, diffraction, elasticityan steady state heat eqation [9-11]. In the last few years some papers involving nonstatianary heat eqation with application of dual integral were published [1,2,6]. In this paper, we will discuss the solution of heat equation related to moving solid heat object with the use of opertional calculus method and dual integral equations with kernel of a Bessl function of the first kind of order zero and unknown function, weight function, free term with dependence of a Laplae transform parameter.The dual integral equations were reduced to a Fredholm integral equation of the second kind by using some known discontinious integrals.

## 2. Mathematical Formulation Of The Problem

Consider a prccess of heat body moves along $x-axis$ with velocity $v$ on a semi-space $z>0$ in a cartesian coordinates $xyz$ of a solid object., with a heat source moves inside a disk $x^2+y^2<b^2$, [3] . Outside the disk $x^2+y^2>b^2$, a temperature function is given. Find the temperature distribution of this body.

It is it required to solve a non-stationary heat conductivity differential equation

$$\nabla^2 T-\frac{v}{a}\frac{\partial T}{\partial x}=\frac{1}{a}\frac{\partial T}{\partial \tau}, v>0, \tau>0, a>0, \tag{2.1}$$

$\nabla^2$ is a Laplaces operator , $T=T(x,y,z)$, where $x,y,z$ cartesian coordinates, The heat coefficient $a$ is independent of a temperature or coordinates (constant). Use the substitution $T=u\exp(-wx), w=v/2a$, equation (2.1) should be written as [3]

$$\nabla^2 u-w^2 u=\frac{1}{a}\frac{\partial u}{\partial \tau}, \tag{2.2}$$

$u=u(x,y,z)$. $\nabla^2 u$ in cylindrical coordinates should be written as

$$\nabla^2 u=[\partial^2/\partial z^2+\frac{1}{r}\partial/\partial r(r\partial/\partial r)]u$$

The initial condition

$$u(r,z,0)=T(r,z,0)-T_0=0 \tag{2.3}$$

The boundary conditions are

$$\left.\frac{\partial u}{\partial r}\right|_{r=0}=\left.\frac{\partial u}{\partial r}\right|_{r\to\infty}=\left.u\right|_{z\to\infty}=\left.\frac{\partial u}{\partial z}\right|_{r\to\infty}=0 \tag{2.4}$$

and under the mixed discontinuous boundary conditions along the level surface $z=0$

$$\frac{\partial u}{\partial z}=-m_1(r,\tau) \qquad ,r\in(0,b) \tag{2.5}$$

$$u=m_2(r,\tau), r\in(b,\infty) \tag{2.6}$$

$m_1(r,\tau)$ is a heat flux (heat source) obey Newtons low of heating inside the disk $r<b$, $m_2(r,\tau)$ is a temperature function acted outside the disk $r>b$. The known functions $m_i(r,\tau), i=1,2$ continuous and have the limited variation with respect of each of the variables r and $\tau$, moreover [2]

$$\int_0^\infty |m_i(r,\tau)|dr < \infty \quad \int_0^\infty |m_i(r,\tau)|d\tau < \infty \ , \ i = 1,2.$$

These restrictions allow to apply Laplace transform with respect to $\tau$ and Hankle transform with respect to $r$ moreover, we assume that the functions $m_i(r,\tau)$, $i = 1,2$ have absolutely continuous derivative with respect to $r$ .

Applying a Laplace transform to (2.2)-(2.6), where

$$\overline{u}(r,z,s) = \int_0^\infty u(r,z,\tau)\exp(-s\tau\ )d\tau$$

A general solution of the given problem becomes

$$\overline{u}(r,z,s) = \int_0^\infty \overline{A}(p,s)\exp(-z\gamma(p,s))J_0(pr)dp\ , \qquad (2.7)$$

$$\gamma(p,s) = \sqrt{p^2 + (w^2 + s/a)}\ .$$

Use mixed boundary conditions (2.5) and (2.6) in the Laplace transform image, we get the pair of dual integral equations to find the unknown function $\overline{A}(p,s)$

$$\int_0^\infty \overline{A}(p,s)\ \gamma(p,s)J_0(pr)dp = \overline{m}_1(r,s),\ r \in (0,b) \qquad (2.8)$$

$$\int_0^\infty \overline{A}(p,s)J_0(pr)dp = \overline{m}_2(r,s), \qquad r \in (b,\infty) \qquad (2.9)$$

At $w \to 0$, the solution of (2.7),(2.8) tends to known results [1]

To solve (2.7),(2.8), let us to express a function $\overline{m}_2(r,s)$ as

$$\overline{m}_2(r,s) = \int_0^\infty \overline{G}(y,s)J_0(pr)dp \qquad (2.10)$$

$\overline{G}(p,s)$ is a known function, applying the inverse Hankel transform to expression (2.9)

$$\overline{G}(p,s) = \int_0^\infty yp\ \overline{m}_2(y,s)J_0(py)dy\ .$$

Next, take $\overline{B}(p,s) - \overline{G}(p,s) = \overline{A}(p,s)$, the dual integral equations were obtained to determine $\overline{B}(p,s)$

$$\int_0^\infty \overline{B}(p,s)\ \gamma(p,s)J_0(pr)dp = \overline{M}(r,s),\ r \in (0,b) \qquad (2.11)$$

$$\int_0^\infty \overline{B}(p,s) J_0(pr)dp = 0, \quad r \in (b,\infty) \tag{2.12}$$

$$\overline{M}(r,s) = \overline{m}_2(r,s) - \int_0^\infty \sqrt{p^2+d}\ \overline{G}(p,s)\ J_0(pr)dp$$

To solve, (2.10), (2.11), replace $\overline{B}(p,s)$ by another unknown function with help of the relation

$$\overline{B}(p,s) = \int_0^b \overline{\phi}(t,s)\Gamma(t,p)dt\ , \tag{2.13}$$

$$\Gamma(t,p) = \frac{p}{\sqrt{p^2+d}} \sin(t\sqrt{p^2+d}\ ),\ d = w^2 + s/a$$

use the discontinuous integrals[ 4]

$$\int_0^\infty \frac{p\,J_0(pr)}{\sqrt{p^2+d}} \sin\left(t\sqrt{p^2+d}\right)dp = \begin{cases} 0 & r > t \\ \dfrac{\cos\sqrt{(t^2-r^2)d}}{\sqrt{t^2-r^2}} & t > r \end{cases} \tag{2.14}$$

$$\int_0^\infty J_1(pr)\sin\left(t\sqrt{p^2+d}\right)dp =$$

$$\frac{1}{r}\begin{cases} \sin(t\sqrt{d}) - \dfrac{-t}{\sqrt{t^2-r^2}}\sin\left(\sqrt{(t^2-r^2)d}\ \right), & 0 < r < t < b, \\ \sin(t\sqrt{d}) - \dfrac{t}{\sqrt{r^2-t^2}}\exp\left(-\sqrt{(r^2-t^2)d}\ \right), & 0 < t < r < b. \end{cases} \tag{2.15}$$

Substitution expression (2.12) into (2.11) and use (2.13), ensure the equality zero. Substitution (2.12) into relation (2.10),use some discontinuous integrals(2.15), we get a first kind singular integral equations to determine $\overline{\psi}(t,s)$

$$\int_0^r \frac{ch\sqrt{(r^2-t^2)d}}{\sqrt{r^2-t^2}}\overline{\psi}(t,s)dt = \overline{M}(r,s)$$

$$+\int_0^b \left\{\frac{\sin\sqrt{(t^2-r^2)d}}{\sqrt{t^2-r^2}} - \frac{\sin(\sqrt{d}\,t)}{t}\right\}\overline{\psi}(t,s)dt \quad r \in (0,b) \tag{2.16}$$

Where $\overline{\psi}(t,s) = t\,\overline{\phi}(t,s)$. Treating (2.16) as an Abels integral equation, a Fredholm integral equation of the second kind is obtained with the unknown function $\overline{\psi}(t,s)$

$$\overline{\psi}(t,s)+\int_0^R \overline{\psi}(\xi,s)\overline{K}(\xi,s,t)dt=\overline{F}(t,s) \quad , \quad r\in(0,b) \qquad (2.17)$$

the free term and the kernel

$$\overline{F}(t,s)=\frac{2}{\pi}\frac{d}{dt}\int_0^t y\,\frac{\cos\sqrt{(t^2-y^2)d}}{\sqrt{t^2-y^2}}\overline{M}(y,s)dy \qquad (2.18)$$

$$\overline{K}(t,\xi,s)=\frac{2}{\pi}\frac{d}{dt}\int_0^t y\,\frac{\cos\sqrt{(t^2-y^2)d}}{\sqrt{t^2-y^2}}\left(\frac{\sin\sqrt{(\xi^2-y^2)d}}{\sqrt{\xi^2-y^2}}-\frac{\sin(\sqrt{d}\,\xi)}{\xi}\right)dy \qquad (2.19)$$

(2.18) and (2.19) should be satified

$\int_0^b \left|\overline{F}(r,s)\right|dr<\infty$ and $\int_0^b\int_0^b \left|\overline{K}^2(r,t,s)\right|dr<\infty$ [7]

Integral equation (2.17) should be be solved numerically or by successive approximation techniques by expanding $\sin(x),\cos(x)$ in appropriate Maclaurin series with the help of some math software [5,7] .The inverse Laplace tansform exists for (2.19) by multiplying the left and right sides of the equation by $\exp(-s),\mathrm{Re}\,s>0$. If $w\to 0$, we can use the expression

$$\overline{\psi}(t,s)=\sum_{m=-\infty}^{-1}\psi_m(t)s^{m/2}+\exp(-bk)\sum_{m=0}^{\infty}\psi_m(t)s^{\frac{m}{2}-1}, \quad k^2=s/a$$

if we substitute the last expression in (2.19), we recieve a recurrent formula for determination $\psi_m(t), n=0,1,2,\ldots$ [1]. Above theory involving dual integral equation can be used to solve several problems of mathematical physics equations involving heat conduction equation.

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