Applied Mathematical Sciences, Vol. 3, 2009, no. 19, 941 - 947

A Note on the Derivation of Fréchet and Gâteaux

Oswaldo González-Gaxiola[1]

Departamento de Matemáticas Aplicadas y Sistemas
Universidad Autónoma Metropolitana-Cuajimalpa
Artificios 40, Col. Miguel Hidalgo, Delegación Álvaro Obregón
México, D. F., C.P. 01120. México

Abstract

The purpose of this note is in addition to establishing Fréchet derivatives and Gâteuax, considering the basic different implications between them. Also be considered a counterexample of a Lipschitzian real-valued function Gâteaux differentiable but not Fréchet differentiable.

Mathematics Subject Classification: Primary 28A10; Secondary 26A16

Keywords: Fréchet differentiability, Gâteaux differentiability, Lipschitzian functions, Banach space.

1 Introduction

In differential analysis, the Fréchet derivative is defined on Banach spaces and the Gâteaux derivative is a generalization of the directional derivative studied extensively in several variables. In general, we know that a function defined between normed spaces differentiable in the sense of Fréchet then it is Gâteaux

[1]ogonzalez@correo.cua.uam.mx

differentiable, the reciprocal is not true in general as seen in some examples shown in [2], [3]. In this note we will study a Lipschitzian real-valued function Gâteaux differentiable but not Fréchet differentiable.

2 The Fréchet and Gâteaux differential

Throughout this work, $\mathbb{E}$ and $\mathbb{F}$ will denote Banach spaces over the real or complex field. In contexts in which two o more spaces appear (for example, in expressions like $\mathcal{L}(\mathbb{E},\mathbb{F})$) they will be understood to be over the same field, although occasional use will be made of the elementary fact that a linear space over the complex field can be considered as a linear space over the real field.

A function f from a set $A \subseteq \mathbb{E}$ into $\mathbb{F}$ is said to be Fréchet differentiable at a if a is an interior point of A and there exists $L \in \mathcal{L}(\mathbb{E},\mathbb{F})$ such that

$$\lim_{x\to a}\frac{f(x)-f(a)-L(x-a)}{||x-a||}=0. \tag{1}$$

This limit relation is, of course, equivalent to the statement that

$$f(x)=f(a)+L(x-a)+o(||x-a||) \quad \text{as} \quad x\to a. \tag{2}$$

We observe that the linear term on the right side of (2) is of larger order than the last term except when $L=0$, for if $L(x-a)/||x-a|| \to 0$, then for any non-zero $x\in\mathbb{E}$

$$L(x)=||x||\frac{L(tx)}{||tx||}=||x||\frac{L(tx+a-a)}{||tx+a-a||}\to 0, \quad \text{as } t\to 0^{+},$$

so that $L=0$.

The statement that f is Fréchet differentiable at a therefore means that it is possible to approximate to $f(x)$ in the neighborhood of a by an expression of the form $f(a)+L(x-a)$, with $L\in\mathcal{L}(\mathbb{E},\mathbb{F})$, to the degree of approximation prescribed by (2).

If f is Fréchet differentiable at a, then the (unique) function $L \in \mathcal{L}(\mathbb{E}, \mathbb{F})$ determined by (1) or (2) is called the Fréchet differential of f at a and we denote it by $df(a)$.
If A is an open set in $\mathbb{E}$, a function f which is Fréchet differentiable at each point of A, it said to be Fréchet differentiable on A. We say also that f is $C^1(A)$ if df is continuous on A.

We note that it is often more convenient to write the limit realtion (1) as

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0, \tag{3}$$

and similarly for (2).

Definition 2.1. *A function f from an open set $A \subseteq \mathbb{E}$ into $\mathbb{F}$ is said Gâteaux differentiable at $a \in A$ if for all $v \in \mathbb{E}$ if there is the limit*

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t},$$

which denote by $\partial f(a, v)$.

Proposition 2.2. *Let $\mathbb{E}$ and $\mathbb{F}$ be normed spaces; $A \subseteq \mathbb{E}$ is an open set and $a \in A$, $f : A \to \mathbb{F}$ Fréchet differentiable at a, then for all $v \in \mathbb{E}$,*

$$df(a)(v) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}.$$

i.e. f is Gâteaux differentiable at a.

Proof Suppose $v \in \mathbb{E}$, $v \neq 0$, as $a \in A$ is an interior point exist $\delta > 0$ such that if $t \in \mathbb{R}$ with $||tv|| = |t|||v|| < \delta$ then $a + tv \in A$ and we obtain

$$f(a+tv) = f(a) + df(a)(tv) + r(tv), \quad \text{where } \lim_{t \to 0} \frac{r(tv)}{||tv||} = 0.$$

Now, for $t \neq 0$, we obtain

$$\frac{f(a+tv) - f(a)}{t} - \frac{r(tv)}{t} = df(a)(v).$$

The right side of equality exists independently of t and therefore

$$\lim_{t\to 0}\left\{\frac{f(a+tv)-f(a)}{t}-\frac{r(tv)}{t}\right\}=df(a)(v),$$

finally as

$$\lim_{t\to 0}\frac{r(tv)}{t}=||v||\lim_{t\to 0}\frac{r(tv)}{t||v||}=\pm||v||\lim_{t\to 0}\frac{r(tv)}{||tv||}=0.$$

Then the proposition is proven.

The reciprocal of the proposition is not necessarily true as we see in [2]. We will now have another counterexample that the reciprocal is false in general (even when the function is Lipschitz).

3 A function Gâteaux differentiable but not Fréchet differentiable

We consider two functions:

$$f: L^1([0,\pi])\to\mathbb{R}\quad\text{defined by } f(x)=\int_0^\pi \sin x(t)dt, \tag{4}$$

and

$$g: L^2([0,\pi])\to\mathbb{R}\quad\text{defined by } g(x)=\int_0^\pi \sin x(t)dt. \tag{5}$$

Clearly, g is the restriction of f to $L^2([0,\pi])\subseteq L^1([0,\pi])$. In order to consider the differentiability of f and g, let $x,\ v\in L^1([0,\pi])$, $v\neq 0$, $h>0$. Then[2]

$$\lim_{t\to 0}\frac{1}{h}\int_0^\pi[\sin(x(t)+hv(t))-\sin x(t)]dt=\int_0^\pi v(t)\cos x(t)dt.$$

Hence, the Gâteaux derivative of the function f at x is $\cos x$. The function g, which is the restriction of f to $L^2([0,\pi])$, is also Gâteaux differentiable and the Gâteaux derivative $\partial g(x,\cdot)=\cos x$ is a continuous function from $L^2([0,\pi])$ in itself in the norm topologies. Therefore, g is Fréchet differentiable everywhere;

[2] $((\sin\frac{hv(t)}{2})/(\frac{h}{2}))\cos(x(t)+\frac{hv(t)}{2})$ is dominated by $v\in L^1([0,\pi])$

see [1]. Actually, in our case, g is uniformly Fréchet differentiable.
Now to prove that f is not Fréchet differentiable and we will follow Sova's proof of [6]; but with the difference that the proof will be done in $L^1([0,\pi])$ and not in $L^2([0,\pi])$. We consider any point $x \in L^1([0,\pi])$ and we will be proof for that x exists $v \in L^1([0,\pi])$ such that the Lebesgue measure of the set

$$\{t \in \mathbb{R} : 0 \leq t \leq \pi \; and \; \sin(x(t)+v(t)) - \sin x(t) - \sin v(t) \cos x(t) \neq 0\}$$

is positive. If not, let $r \in \mathbb{Q}$ (The set of rational numbers) and define v_r by

$$v_r(t) = \begin{cases} r, \text{ if } t \in [0,\pi] \\ 0, \text{ if } t \in \mathbb{R} \setminus [0,\pi]. \end{cases}$$

Then $v_r \in L^1([0,\pi])$ and the set

$$M_r = \{t \in [0,\pi] : \sin(x(t)+r) - \sin x(t) \neq r \cos x(t)\}$$

has Lebesgue measure zero. Hence, the set $M = \cup_{r \in \mathbb{Q}} M_r$ also has measure zero. Thus, for all rational numbers r and for all $t \notin M$, we have

$$\sin(x(t)+r) - \sin x(t) = r \cos x(t).$$

This is a contradiction since the function $r \cos x(t)$ is linear of r, but the function $\sin(x(t)+r) - \sin x(t)$ in not linear.
Now, we choose $v_+ \in L^1([0,\pi])$ such that

$$\mu(\{t \in [0,\pi] : \sin(x(t)+v_+(t)) - \sin x(t) - v_+(t) \cos x(t) \neq 0\}) > 0,$$

where μ denotes Lebesgue measure. Then we can find $\alpha > 0$ such that the set

$$K_\alpha = \{t \in [0,\pi] : \sin(x(t)+v_+(t)) - \sin x(t) - v_+ \cos x(t) > \alpha\}$$

has $\mu(K_\alpha) > 0$. Moreover, there exists a $\beta > 0$ and a measurable subset $K_{\alpha,0}$ of K_α such that $\mu(K_{\alpha,0}) > 0$ and $|v_+(t)| < \beta$ for $t \in K_{\alpha,0}$. Choose a decreasing sequence $\{K_n\}_{n=1}^{\infty}$ of measurable subsets of $K_{\alpha,0}$ of K_α such that $\mu(K_n) > 0$

for all $n \in \mathbb{N}$ and $\cap_{n=1}^{\infty} K_n = \emptyset$, and define a sequence $\{\psi_n\}_{n=1}^{\infty}$ of functions in $L^1([0,\pi])$ by

$$\psi_n(t) = \begin{cases} v_+(t), & \text{if } t \in K_n \\ 0, & \text{if } t \notin K_n. \end{cases}$$

We can easily check that $||\psi_n||_1 \to 0$ as $n \to \infty$, but

$$\frac{|\int_0^{\pi}[\sin(x(t)+\psi_n(t)) - \sin x(t) - \psi_n(t)\cos x(t)]dt|}{||\psi_n||_1} \geq \frac{\alpha\mu(K_n)}{\beta\mu(K_n)} = \frac{\alpha}{\beta} > 0.$$

This shows that f is not Fréchet differentiable at $x \in L^1([0,\pi])$. We can easily see that f is a Lipschitzian real-valued function Gâteaux differentiable but not Fréchet differentiable, see [2], [4].

ACKNOWLEDGEMENTS. I am grateful to the departments of mathematics of the Universidad Autónoma Metropolitana-Cuajimalpa and Iztapalapa for the valuable help to write this paper.

References

[1] J. P. Aubin, I. Ekeland *Applied nonlinear analysis,* Wiley , New York (1984).

[2] D. Behmardi , E. D. Nayeri, *Introduction of Fréchet and Gâteaux Derivative,* Applied. Math. Sciences. Vol. 2, No. 20 (2008), 975-980.

[3] J. F. Caicedo, *Cálculo Avanzado (in spanish),* Univ. Nal. de Colombia, Bogotá, 2005.

[4] R. Deville, P Hájek *On the range of the derivative of Gâteaux-smooth functions on separable Banach spaces,* Israel Journal of Mathemmatics. 145, (2005), 257-269.

[5] T. M. Flett, *Differential Analysis,* Cambridge Univ. Press, Cambridge, (1980).

[6] M. Sova, *Conditions for differentiability in linear topological spaces,* Czechoslovak. Math. J. 16, (1966), 339-362.

Received: June, 2009