

On the Convolution Order with Reliability Applications

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Abstract

In this paper, we study further the convolution order and investigate its reliability properties. We apply a simple useful property of this order to families of non-negative random variables that have the generalized semigroup property. Some characterizations and applications in reliability theory are described.

Keywords: Convolution order, Laplace transform order, Laplace transform ratio order, shock models, generalized semi-group property, self-decomposable distributions

1 Introduction

Stochastic comparisons between probability distributions play a fundamental role in probability, statistics and some related areas, such as reliability theory, survival analysis, economics and actuarial science (see, Muller and Stoyan (2002) and Shaked and Shanthikumar (2004) for an exhaustive monograph on this topic).

One of these comparison is the Laplace transform order (\leq_{Lt}), that we recall here. Throughout this paper, X and Y are two random variables with distribution functions F_X and F_Y , respectively. Denote by L_X the Laplace transform of F_X , and by $\overline{F}_X = 1 - F_X$ the corresponding survival function. We use a similar notation for all other distribution functions. Moreover, we will

use the term *decreasing* in place of *non-increasing*. Also, it is always assumed in the sequel that $\stackrel{d}{=}$ denotes equality in distribution.

For a non-negative random variables X , the Laplace-Stieltjes transform of F_X is given by

$$L_X(s) = \int_0^{\infty} e^{-su} dF_X(u), \quad s > 0.$$

A non-negative random variable X is said to be smaller than Y in the Laplace transform order (denoted by $X \leq_{Lt} Y$) if, and only if, $L_X(s) \geq L_Y(s)$, for all $s > 0$. Let us observe that if we denote

$$\begin{aligned} L_X^*(s) &= \int_0^{\infty} e^{-su} \bar{F}_X(u) du \\ &= \frac{1 - L_X(s)}{s}, \end{aligned}$$

(1.1)

then $X \leq_{Lt} Y$ if, and only if, $L_X^*(s) \leq L_Y^*(s)$, for all $s > 0$.

Applications, properties and interpretations of the Laplace transform order can be found in Klefsjo (1983), Alzaid et al. (1991), Shaked and Wong (1997), Belzunce et al. (1999), Gao et al. (2002), Ahmed and Kayid (2004) and Ahmad and Kayid (2005).

Another comparisons based on the Laplace–Stieltjes transforms of the distributions of X and Y has been defined and studied in Shaked and Wong (1997). A non-negative random variable X is said to be smaller than Y in the Laplace transform ratio order (denoted by $X \leq_{Lt-r} Y$) if, and only if,

$$\frac{L_Y(s)}{L_X(s)} \text{ is decreasing in } s > 0.$$

In the current investigation, we further develop a stronger notion of stochastic orders of non-negative random variables via ratios that are determined by their Laplace transforms, which called the convolution order, and investigate its reliability properties. The definition, as well as some characterizations of this order will be given in Section 2 below. In that section we apply a simple useful property of the convolution order to families of non-negative random variables that have the generalized semigroup property. In Section 3 a few applications in reliability theory are described.

1.1 The convolution order

First, recall that a non-negative function ϕ is a Laplace transform of a non-negative measure on $(0, \infty)$ if, and only, if ϕ is completely monotone (*c.m.*): that is, all of the derivatives $\phi^{(n)}$ of ϕ exist, and they satisfy $(-1)^n \phi^{(n)}(x) \geq 0$, for $x > 0$ and $n = 1, 2, \dots$, where $\phi^{(0)} \equiv \phi$. For two non-negative random variables X and Y , if

$$\frac{L_Y(s)}{L_X(s)} \text{ is } c.m. \text{ in } s > 0,$$

then we say that X is smaller than Y in the convolution order (denoted by $X \leq_{conv} Y$). Some characterizations and preservation properties of the convolution order can be found in Shaked and Suarez-ILorens (2003). From (1.1) it is seen that

$$X \leq_{conv} Y \Leftrightarrow \frac{1 - sL_Y^*(s)}{1 - sL_X^*(s)} \text{ is } c.m. \text{ in } s > 0.$$

Some basic properties of the convolution order are given below. The next lemma gives a simple useful property of the convolution order. We apply this property to families of non-negative random variables that have the generalized semigroup property.

Lemma 2.1.

Let X and Y be two non-negative random variables. Then $X \leq_{conv} Y$ if, and only if, there exists a non-negative random variable Z independent of X , such that

$$Y \stackrel{d}{=} X + Z.$$

Proof.

Since $X \leq_{conv} Y$ if, and only if, $L(t) = L_Y(t)/L_X(t)$ is *c.m.* function and $L(0) = 1$, it follows that $L(t)$ is a Laplace transform of a distribution function on $(0, \infty)$. By extending the probability space in which X is defined, we can define a random variable Z independent of X with Laplace transform equal to $L(t)$ and hence the result follow. ■

The convolution order sometimes can be used as a realistic assumption in some statistical inferential applications. For instance, consider the problem of the nonparametric estimation of two life distributions, F and G , in two-sample problem. Suppose that any (observed) lifetime of interest here is a sum of two

(unobserved) independent non-negative random variables (generically denoted by Z) is determined by the environment in which the lifetime is observed, and the distribution of the second random variable (generically denoted by X) is independent of that environment. In such a case, if F is the distribution of the lifetimes observed in a certain environment in which Z is essentially 0, and G is the distribution of lifetimes observed in another environment in which Z is positive, then it is realistic to assume that $F \leq_{conv} G$. (Here, the notation $F \leq_{conv} G$ means that the two underlying random variables are ordered with respect to \leq_{conv}).

The convolution order imply the usual stochastic order (for the definition of the usual stochastic order see, Shaked and Shanthikumar 2004), and the usual stochastic order is known to imply the Laplace transform order. Also, it is seen that

$$X \leq_{conv} Y \Rightarrow X \leq_{Lt-r} Y \Rightarrow X \leq_{Lt} Y, \quad (2.3)$$

Another useful order is the hazard rate order. A non-negative random variable X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if, and only if, $\bar{F}_X(s)/\bar{F}_Y(s)$ is decreasing in $s > 0$. The following proposition gives the relation between the hazard rate order and the convolution order.

Proposition 2.1.

Let X^* and Y^* be random variables with distribution functions given by a mixture of exponential distributions with means $1/x$, $x > 0$, and a mixing distribution F_X and F_Y , respectively. If $X \leq_{conv} Y$ then $Y^* \leq_{hr} X^*$.

Proof.

First, note that the distribution functions given by a mixture of exponential distributions with means $1/x$, $x > 0$, and a mixing distribution F may be written as

$$\begin{aligned} F^*(s) &= \int_0^\infty (1 - e^{-sx}) dF(x), \quad s \geq 0 \\ &= 1 - L_X(s). \end{aligned}$$

Also, it is easy to notice that

$$X \leq_{Lt-r} Y \Leftrightarrow Y^* \leq_{hr} X^*,$$

The result now follows from (2.3). ■

The next result gives a preservation property for the \leq_{conv} under random sum. Such results have applications in reliability theory, as will be seen in Section 3.

Theorem 2.1.

Let X_1, X_2, \dots , be independent and identically distributed non-negative random variables and let N_1 and N_2 be positive integer-valued random variables which are independent of the X_i 's. Then

$$N_1 \leq_{conv} N_2 \Leftrightarrow \sum_{i=1}^{N_1} X_i \leq_{conv} \sum_{i=1}^{N_2} X_i.$$

Proof.

Let $L_{X_1}(t)$ and $L_{X_2}(t)$ be the Laplace transforms of $\sum_{i=1}^{N_1} X_i$ and $\sum_{i=1}^{N_2} X_i$ respectively. We have

$$\frac{L_{X_2}(t)}{L_{X_1}(t)} = \frac{L_{N_2}(-\log L_{X_2}(t))}{L_{N_1}(-\log L_{X_1}(t))}.$$

Therefore the result follows from Lemma 2.1 and from the assumptions. ■

1.2 Classes ordered by the convolution order

In this subsection, we apply the convolution order to families of random variables of the form $\{X(\theta), \theta \in \Theta\}$. This notation stands for a collection of random variables with distribution functions parametrized by θ . That is, these random variables are associated with a family $\{F_\theta, \theta \in \Theta\}$ of univariate distribution functions, where F_θ is the distribution function of $X(\theta)$. Note that we are concerned here only with the marginal distributions of the $X(\theta)$'s, even if in some applications $\{X(\theta), \theta \in \Theta\}$ may be a well-defined stochastic process. In all the applications below Θ will be an interval in $(-\infty, \infty)$ or in $\{\dots, -1, 0, 1, \dots\}$.

Generalized Semi-group classes: First, we give the definition of the semi-group and generalized semi-group classes. The family of random variable $\{X(\theta), \theta \in \Theta\}$ is said to have:

- (a) the semi-group property if for every $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$, such that $\theta_1 < \theta_2$, we have that $\theta_2 - \theta_1 \in \Theta$ and there exist independent random variables Z_1 and Z_2 such that

$$X(\theta_2) \stackrel{d}{=} Z_1 + Z_2, \quad Z_1 \stackrel{d}{=} X(\theta_1), \quad Z_2 \stackrel{d}{=} X(\theta_2 - \theta_1).$$

- (b) the generalized semigroup property if for every $\theta_1 \in \Theta$ and $\theta_2 \in \Theta$, such that $\theta_1 < \theta_2$, there exist independent random variables Z_1 and Z_2 such that

$$X(\theta_2) \stackrel{d}{=} Z_1 + Z_2 \quad \text{and} \quad Z_1 \stackrel{d}{=} X(\theta_1). \quad (2.4)$$

Some examples of families of random variables that have the generalized semigroup property can be found in Shaked et al. (1995) and Di Crescenzo and Shaked (1997). It should be pointed out that the above definition of the generalized semigroup property is slightly different than Definition 2.8(ii) of Shaked et al. (1995). However, all examples in Remark 2.9 of Shaked et al. (1995) satisfy the above definition. Note that if a family of random variables $\{X(\theta), \theta \in \Theta\}$ satisfies the semigroup property then it also satisfies the generalized semigroup property. Shaked et al. (1995) also introduced definitions of sub-semigroup and super-semigroup properties. Again, if a family of random variables satisfies the sub-semigroup or the super-semigroup property then it also satisfies the generalized semigroup property. The results below hold for families that have the generalized semigroup property, and therefore they hold also for families that have the semigroup, the sub-semigroup, or the super-semigroup properties.

Using Lemma 2.1 the following result can be easily proven; it will be the key for the applications that follow.

Theorem 2.2.

Let $\{X(\theta), \theta \in \Theta\}$ be a family of non-negative random variables that have the generalized semigroup property such that, for any $\theta_1 < \theta_2$, the random variable Z_2 of (2.4) is non-negative *a.s.* Then

$$X(\theta_1) \leq_{conv} X(\theta_2) \quad \text{for} \quad \theta_1 < \theta_2.$$

Proof.

Since $X(\theta_1) \stackrel{d}{=} Z_1$ and $X(\theta_2) \stackrel{d}{=} Z_1 + Z_2$, where Z_1 and Z_2 are independent and non-negative, the stated result follows from Lemma 2.1. ■

Some examples of classes of distributions that have the generalized semigroup property are the following.

Example 2.1.

Let $X(\theta)$ be a Poisson random variable with mean $\mu(\theta) > 0$ that increasing in θ . Then $\{X(\theta), \theta \in \Theta\}$ has the generalized semigroup property and it satisfies the conditions of Theorem 2.2. Therefore

$$X(\theta_1) \leq_{conv} X(\theta_2), \text{ for } \theta_1 < \theta_2.$$

Example 2.2.

Let $X(\theta)$ be a gamma random variable with a fixed scale parameter and with shape parameter $\alpha(\theta) \geq 0$ that is increasing in θ . Then $\{X(\theta), \theta \in \Theta\}$ has the generalized semigroup property and it satisfies the conditions of Theorem 2.2.

Example 2.3.

For $\theta \in \{1, 2, \dots\}$, and a fixed $p \in (0, 1)$, let $X(\theta)$ be a negative binomial random variable with probabilities

$$\Pr \{X(\theta) = x\} = \binom{x-1}{\theta-1} p^\theta (1-p)^{x-\theta}, \quad x = \theta, \theta+1, \dots .$$

Then $\{X(\theta), \theta \in \{1, 2, \dots\}\}$ has the semigroup property and it satisfies the conditions of Theorem 2.2.

Example 2.4.

For $\theta \in \{1, 2, \dots\}$, and a fixed $p \in (0, 1)$, let $X(\theta)$ be a binomial random variable with probabilities

$$\Pr \{X(\theta) = x\} = \binom{\theta}{x} p^x (1-p)^{\theta-x}, \quad x = 0, 1, \dots, \theta.$$

Then $\{X(\theta), \theta \in (1, 2, \dots)\}$ has the semigroup property and it satisfies the conditions of Theorem 2.2.

Self-decomposable distributions: Another useful class is the one formed by self-decomposable distributions. A non-negative random variable X is said to be self-decomposable if for all $0 < \rho < 1$, there exists a (innovation) random variable Z_ρ such that

$$X \stackrel{d}{=} \rho X + Z_\rho.$$

Such a random variable X is infinitely divisible and unimodal (see, e.g. Vervaat (1979) and Yamazato (1978)). Assuming X is non-negative, it follows X is self-decomposable if, and only if for all $0 < \rho < 1$

$$\rho X \leq_{conv} X.$$

It is well known that only continuous random variables satisfy the above definition of self-decomposability. For a non-negative integer-valued random variable X , let $\alpha \circ X = \sum_{i=1}^X B_i$ where $\{B_i\}$ is a sequence of independent and identically distributed Bernoulli random variables with parameter α , that are independent of X . Then X is discrete self-decomposable if

$$X \stackrel{d}{=} \alpha \circ X + Z_\alpha,$$

where Z_α is a random variable independent of $\alpha \circ X$ for all $\alpha \in (0, 1)$ (see, Stuetal and Van-Harn (1979)). Therefor it follows that X is discrete self-decomposable if, and only if for all $0 < \alpha < 1$

$$\alpha \circ X \leq_{conv} X.$$

2 Reliability applications

Reliability theory is a body of ideas, mathematical models, and methods aimed at predicting, estimating, understanding, and optimizing the life span and failure distributions of systems and their components (Barlow and Proschan, 1981)). Also, it allows researchers to predict the age-related failure kinetics for a system of given architecture (reliability structure) and given reliability of its components. In this section we provide few applications of the convolution order in reliability theory.

Application 3.1 (Shock models)

Shock models are of great interest in the context of reliability theory. The class of shock models arises in reliability theory when a system or an item is subjected to random shocks over time. The system is assumed to have an ability to withstand a random number of these shocks, and it is commonly assumed that the number of shocks and the interarrival times of shocks are independent. If we denote by N the number of shocks survived by the system and by E_j the random interarrival time between the $(j-1)$ -th and j -th shocks, then the lifetime T of the system is given by $T = \sum_{j=1}^N E_j$. Therefore shock models are particular cases of random sums. In particular, if the interarrivals are assumed to be independent and exponentially distributed (with common parameter λ), then the survival function of T can be written as

$$\bar{H}(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda x} (\lambda x)^k}{k!} P(N > k), \quad x \geq 0.$$

Shock models of this kind, called Poisson shock models, have been studied extensively in the literature. Some recent references are Alzaid et al. (1991), Pellerey (1994), Fagioli and Pellerey (1994), Shaked and Wong (1997) and Belzunce et al. (1999) among others. Therefore as an application of Theorem 2.1 gives the following result.

Theorem 3.1.

Let N_1 and N_2 be two positive integer-valued random variables and, for $j = 1, 2$ let T_j have the survival function

$$\overline{H}_j(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda x} (\lambda x)^k}{k!} P(N_j > k), \quad x \geq 0.$$

Then $N_1 \leq_{conv} N_2$ if, and only if, $T_1 \leq_{conv} T_2$.

Another shock model is the damage shock model which can be formulated as follows. Suppose that the i th shock causes a random damage E_j exponentially distributed with rate $\lambda > 0$, $i = 1, 2, \dots$, and that the E_j are mutually independent. When the accumulated damage exceeds a random critical threshold Y , the items fails. Let N be the number of shocks survived by the item. Then the probability function of N is given by

$$Q_N(n) = \int_0^{\infty} \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} f_Y(x) dx.$$

Theorem 3.2.

Let Y_1 and Y_2 be two non-negative random variables and, for $j = 1, 2$ let N_j have the above probability function with Y replaced by Y_j .

(i) If $Y_1 \leq_{conv} Y_2$ for fixed $\lambda > 0$ then $N_1 \leq_{conv} N_2$.

(ii) If $N_1 \leq_{conv} N_2$ for all $\lambda > 0$ then $Y_1 \leq_{conv} Y_2$.

Proof.

For $j = 1, 2$, we have

$$L_{N_j}(t) = e^{-t} L_{Y_j}(\lambda(1 - e^{-t})).$$

Suppose $Y_1 \leq_{conv} Y_2$ for fixed λ , then

$$\frac{L_{N_2}(t)}{L_{N_1}(t)} = \frac{L_{Y_2}(\lambda(1 - e^{-t}))}{L_{Y_1}(\lambda(1 - e^{-t}))}.$$

Since $(1 - e^{-t})$ has *c.m.* derivative and $L_{Y_2}(t)/L_{Y_1}(t)$ is *c.m.*, it follows $L_{N_2}(t)/L_{N_1}(t)$ is *c.m.* (Feller, 1971, p. 441).

On the other hand, if $N_1 \leq_{conv} N_2$ for all λ , then we have

$$\begin{aligned} \frac{L_{Y_2}(s\lambda)}{L_{Y_1}(s\lambda)} &= \frac{L_{N_2}(-\log(1-s))}{L_{N_1}(-\log(1-s))} \\ &= G(1-s), \end{aligned}$$

for all $0 < s < 1$ and $\lambda > 0$ where G is a probability generating function. Now the result follows from the fact that G has positive derivatives.

Application 3.2 (First passage times)

Let $\{X(t), t \geq 0\}$ be a continuous-time strong Markov process with state space Θ is an interval of the real line. Suppose that $\{X(t), t \geq 0\}$ has stationary transition probabilities. For $\theta_1, \theta_2 \in \Theta$, such that $\theta_1 < \theta_2$, let X_{θ_1, θ_2} denote the first passage time of the process to θ_2 given that $X(0) = \theta_1$. If, with probability 1, the sample paths of $\{X(t), t \geq 0\}$ have no positive jumps, then

$$X_{\theta_1, \theta_2} \leq_{conv} X_{\theta_1, \theta_3}, \quad \theta_1 \leq \theta_2 \leq \theta_3,$$

and

$$X_{\theta_2, \theta_3} \leq_{conv} X_{\theta_1, \theta_3}, \quad \theta_1 \leq \theta_2 \leq \theta_3.$$

These stochastic inequalities follow from Theorem 3.1 and the facts that for each $\theta_1 \in \Theta$ we have $\{X_{\theta_1, \theta_2}, \theta \in \Theta \cap (\theta_1, \infty)\}$ have the generalized semigroup property; and that for each $\theta_3 \in \Theta$ we have that $\{X_{-\theta, \theta_3}, \theta \in (-\Theta) \cap (-\theta_3, \infty)\}$ have the generalized semigroup property (Marshall and Shaked, 1983).

ACKNOWLEDGMENT

This work was supported by College of Science-Research Center Project No. (Stat/2007/23).

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Received: October, 2008