Explicit Approximate Solution for Finding the

Natural Frequency of the Motion of Pendulum

by Using the HAM

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Abstract

In this work, we use a powerful analytical method called homotopy analysis method (HAM) to derive the frequency of a nonlinear oscillating system. Unlike the perturbation method, the HAM does not require the addition of a small physically parameter to the differential equation. It is applicable to strongly and weakly nonlinear problems. Moreover, the HAM involves an auxiliary parameter, h, which renders the convergence parameter of series solutions Controllable, and increases the convergence, and increases the convergence significantly. This article depicts that the HAM is

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an efficient and powerful method for solving oscillating systems.

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1 Introduction

Modeling of natural phenomena in most sciences yields nonlinear differential equations the exact solutions of which are usually rare. Therefore, analytical methods are strongly needed. For instance, one analytical method, called perturbation, involves creating a small physically parameter in the problem, however, finding this parameter is impossible in most cases [1, 2]. Generally speaking, one simple solution for controlling convergence and increasing it does not exist in all analytical methods.

In 1992, Liao [3] presented homotopy analysis method (HAM) based on fundamental concept of homotopy in topology [4-6]. In this method, we do not need to apply the small parameter and unlike all other analytic techniques, the HAM provides us with a simple way to adjust and control the convergence region of approximate series solutions. HAM has been successfully applied to solve many types of nonlinear problems [7, 8].

2 Basic idea of HAM

In this work, we apply the HAM to obtain the frequency of a nonlinear oscillating system, known as the duffing equation, as follows

$$\ddot{\theta}(t) + \omega_0^2 \sin(\theta(t)) = 0 \ , \tag{1}$$

where t denotes the time, the dot denotes derivative with respect to t, $\omega_0^2 = \frac{g}{l}$ is frequency and $\theta(t)$ is the oscillator displacement. As initial conditions, we take

$$\theta(0) = a \ , \qquad \dot{\theta}_0(0) = 0, \tag{2}$$

The exact frequency is [1]

$$\omega = \frac{2\pi}{T}, \tag{3}$$

$$T = \frac{4}{\omega_0^2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\theta}}, \tag{4}$$

$$k = \frac{|\dot{\theta}_0|}{2\omega_0}, \tag{5}$$

Now we expand $\sin(\theta)$ in (1) around $\theta = 0$ and we obtain

$$\ddot{\theta}(t) + \omega_0^2 (\theta(t) - \frac{\theta^3(t)}{6}) = 0, \tag{6}$$

to account for the nonlinear dependence of the frequency, we explicitly exhibit the frequency ω of the system in the differential equation. To this end, we introduce the transformation.

$$\tau = \omega t, \qquad \theta(t) = u(\tau), \tag{7}$$

Hence, equation (1) becomes

$$\omega^2 u''(\tau) + \omega_0^2 (u(\tau) - \frac{u^3(\tau)}{6}) = 0, \tag{8}$$

$$u(\tau) = a, \quad u'(\tau) = 0, \quad \text{when} \quad \tau = 0. \tag{9}$$

Where the prime indicates the derivative respect to τ. We note that the actual frequency of the system now appears explicitly in the equation. Until now, both u and ω are unknowns. $u(\tau)$ can be expressed in the form

$$u(\tau) = \sum_{n=1}^{+\infty} b_n \cos(n\tau). \tag{10}$$

Where b_n is a coefficient. This provides us with the *so-called solution expression* of $u(\tau)$. $\omega_0^2 = \frac{g}{l}$. According to (10) and (9), it is obvious for us to choose such an initial guess

$$u_0(\tau) = a \cos(\tau), \tag{11}$$

according to Equation (10). We choose the auxiliary linear operator

$$L[\varphi(\tau; p)] = \frac{\partial^2 \varphi(\tau; p)}{\partial \tau^2} + \varphi(\tau; p), \tag{12}$$

which has the property

$$L(c_1 \sin \tau + c_2 \cos \tau) = 0, \tag{13}$$

for any integration constants c_1 and c_2. Now with respect to equation (8) we define the nonlinear auxiliary function N as follows

$$N[\varphi(\tau; p), \Omega(p)] = \Omega^2(p) \frac{\partial^2 \varphi(\tau; p)}{\partial \tau^2} + \Omega^2(0) \varphi(\tau; p) - \frac{\Omega^2(0)}{6} \varphi^3(\tau; p), \tag{14}$$

where $p \in [0,1]$ is an embedding parameter, $\varphi(\tau; p)$ is a kind of mapping of the

unknown function $u(\tau)$ and it is a function of τ and p, $\Omega(p)$ is kind of mapping of unknown frequency ω and it is a function of p.

Now we consider h determine an auxiliary parameter and $H(\tau) \neq 0$ an auxiliary function, respectively. h, increases the results convergence. Then, we construct the so-called *zero-order deformation equation*

$$(1-p)L[\varphi(\tau;p)-u_0(\tau)]=hpN[\varphi(\tau;p),\Omega(p)], \tag{15}$$

subject to the initial conditions

$$\varphi(0;p)=a, \qquad \frac{\partial\varphi(0;p)}{\partial\tau}=0, \tag{16}$$

it is obvious that the solution of equations (15) and (16) for $p=0$ and $p=1$ as follows

$$\varphi(\tau;0)=u_0(\tau), \qquad \Omega(0)=\omega_0, \tag{17}$$

$$\varphi(\tau;1)=u(\tau), \qquad \Omega(1)=\omega. \tag{18}$$

When p increases from zero to one, $\varphi(\tau;p)$ from the $u_0(\tau)=a\cos\tau$ to the $u(\tau)$ of Eqs. (8) and (9) and $\Omega(p)$ from ω_0 to the unknown frequency ω.

Expanding $\varphi(\tau;p)$ and $\Omega(p)$ in Taylor series with respect to p and using (17), we have

$$\varphi(\tau;p)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)p^m, \tag{19}$$

$$\Omega(p)=\omega_0+\sum_{m=1}^{+\infty}\omega_m p^m, \tag{20}$$

$$u_m(\tau)=\frac{1}{m!}\frac{\partial\varphi^m(\tau;p)}{\partial p^m}\bigg|_{p=0}, \tag{21-a}$$

$$\omega_m=\frac{1}{m!}\frac{\partial^m\Omega(p)}{\partial p^m}\bigg|_{p=0}, \tag{21-b}$$

we note that in the zero-order deformation equation $\varphi(\tau;p)$ and $\Omega(p)$ are dependent upon the auxiliary h and auxiliary function $H(\tau)\neq 0$. If the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(\tau)$ are properly chosen so that the series (19) and (20) converge at $p=1$, using (18), then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{22}$$

$$\omega = \omega_0 + \sum_{m=1}^{+\infty} \omega_m. \tag{23}$$

Now for simplicity, we define the vectors of $u_m(\tau)$ and ω_m as follows

$$\overrightarrow{u_m} = \{u_0(\tau), u_1(\tau), ..., u_m(\tau)\},$$

$$\overrightarrow{\omega_m} = \{\omega_0, \omega_1, ..., \omega_m\}.$$

Differentiating the zero-order deformation equation (15) and (18) m times with respect to p, then dividing them by $m!$, and finally setting $p=0$, we have the so-called *mth-order deformation equation* in the following form

$$L[u_m(\tau) - x_m u_{m-1}] = h\, H(\tau) R_m(u_{m-1}, \omega_{m-1}) \tag{24}$$

Subject to initial conditions

$$u_m(0) = 0, \qquad u'_m(0) = 0, \tag{25}$$

where

$$R_m(u_{m-1}, \omega_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(\tau, p), \Omega(p)]}{\partial p^{m-1}} \bigg|_{p=0} = \sum_{k=0}^{m-1} (\sum_{i=0}^{k} \omega_i \omega_{k-i}) u''_{m-1-k} + \omega_0^2 \tag{26}$$

$$u_{m-1} - \frac{\omega_0^2}{6} \sum_{k=0}^{m-1} (\sum_{i=0}^{k} u_i u_{k-i}) u_{m-1-k}$$

where

$$x_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be noted that we assume $H(\tau) = 1$. we should note that both $u_m(\tau)$ and ω_m are unknown. But, we have one equation (24) for $u_m(\tau)$. Then, another algebraic equation must be given so as to determine ω_{m-1}

Regarding the nature my problem we can understand that $R_m(u_{m-1}, \omega_{m-1})$ should be expressed as follows

$$R_m(u_{m-1}, \omega_{m-1}) = \sum_{k=0}^{\mu_m} \beta_{m,k}(\omega_{m-1}) \cos[(2k+1)\tau], \tag{27}$$

where $\beta_{m,k}$ is a coefficient, μ_m is a integer dependent on the *m*.

Regarding to (13), when $\beta_{m,0}(\vec{\omega}_{m-1}) \neq 0$, the solution of *mth*-order (24) disobeys the solution expression (10). Then we should set

$$\beta_{m,0}(\vec{\omega}_{m-1}) = 0, \tag{28}$$

which provides us another algebraic equation for getting ω_{m-1} The *Nth*-order approximation is given as follows

$$u(\tau) \approx u_0(\tau) + \sum_{m=1}^{N} u_m(\tau), \tag{29}$$

$$\omega \approx \omega_0 + \sum_{m=1}^{N} \omega_{m-1}, \tag{30}$$

It should be noted that the HPM is a particular from of the analytical HAM, that is, for $h = -1$ HAM will equal HPM [3].

3 Result Analysis

The objective of this study is to calculate the value of ω, and compare it with exact values in Nayfeh [1]. The auxiliary parameter *h* controls the convergence of series solutions. Once we obtain ω, we will be able to plot ω versus *h* ($\omega - h$ curve), to acquire the appropriate value of *h* to find the exact solution. In this paper, we plot $\omega - h$ curves for $a = 0.0001, l = 1$ after 15 iterations for ω. The optimal *h* for the curve is zero.

The ω values obtained through HAM (after 15, iterations for estimating ω), HPM (after 15, iterations for estimating ω) and the exact solution are shown in table [1]. The solution resulting from HAM coincides with the exact solution. Whereas for $h = -1$ (HPM) the solution is much weaker than those of HAM.

Table 1
The values of ω.

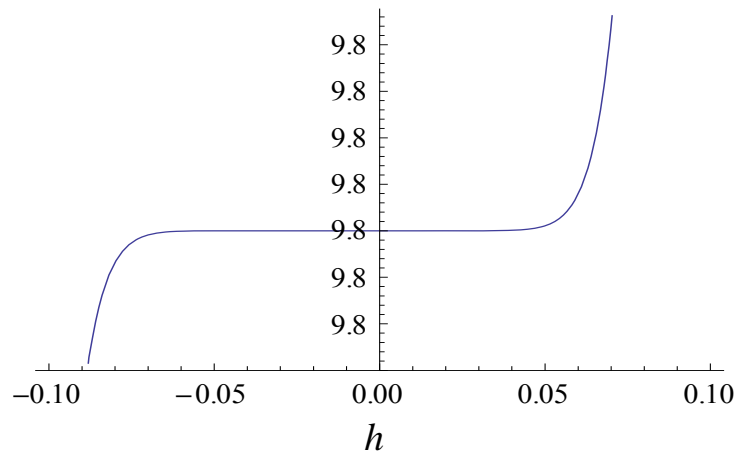| $a = .0001, l = 1$	values
$\omega_{exact} = \dfrac{2\pi}{T_{exact}}$	9.8
ω_{HAM}	9.8
ω_{HPM}	3.2932×10^7

Fig.1. The 15th-order approximation of ω versus h

4 Conclusions

In this paper, we utilized the powerful method of homotopy analysis to obtain the frequency of an oscillating system. We achieved a very good approximation with the actual solution of the considered system. In addition, this technique is algorithmic and it is easy to implementation by symbolic computation software, such as Maple and Mathematica. Different from all other analytic techniques, it provides us with a simple way to adjust and control the convergence region of approximate series solutions. Unlike perturbation method, the HAM does not need any small parameter. It shows that the HAM is a very efficient method. We sincerely hope this method can be applied in a wider range.

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