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Homotopy Perturbation Padé Technique for Constructing Approximate and Exact Solutions of Boussinesq Equations

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Abstract

Based on the homotopy perturbation method (HPM) and Padé approximants (PA), explicit approximate and exact solutions are obtained for cubic Boussinesq equations. HPM is used for analytic treatment to those equations and PA for increasing the convergence region of the HPM analytical solution. The results reveal that the HPM with the enhancement of PA is a very effective, convenient and quite accurate to such types of partial differential equations.

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1. Introduction

The homotopy perturbation method (HPM) was firstly proposed by He [1–4]. It deforms a difficult problem into simple problems that can easily be solved. The HPM, based on a series approximation, is one among the newly developed analytical methods for strongly nonlinear problems and has been proven successful and efficiency in solving a wide class of nonlinear differential equations (NLDEs), various physics and engineering problems [5–7]. But, it's known that convergence region of the obtained truncated series solution is limited and, in great, need enhancements to enlarge region of convergence.

It's well-known that Padé approximations (PA) [8] have the advantage of manipulating the polynomial approximation into a rational function of polynomials. This manipulation provides us with more information about the mathematical behavior of the solution. So, application of PA to the truncated series solution obtained by HPM will be an effective tool to increase the region of convergence and accuracy of the approximate solution even for large values of *t*. In order to obtain Padé approximants of different orders [*N/M*], Maple or Mathematica can be efficiently used.

In this paper, we are interested in applying the HPM with Padé technique to obtain approximate analytical and exact solutions of the cubic Boussinesq and modified Boussinesq equations. Comparison of the present solution is made with the exact one and excellent agreement is noted.

2. Basic idea of He's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [1]:

$$A(u)-f(r)=0, \quad r\in\Omega, \tag{1}$$

with the boundary conditions

$$B(u,\partial u/\partial n)=0, \quad r\in\Gamma, \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be divided into two parts which are L and N, where L is linear, but N is nonlinear. Therefore, (1) can be rewritten as follows:

$$L(u)+N(u)-f(r)=0. \tag{3}$$

By the homotopy technique, we construct a homotopy $V(r,p):\Omega\times[0,1]\to\mathbb{R}$ which satisfies:

$$H(V,p)=(1-p)\left[L(V)-L(u_0)\right]+p\left[A(V)-f(r)\right]=0, \quad p\in[0,1],\ r\in\Omega, \tag{4}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of (1), which satisfies the boundary conditions.

According to the HPM, we assume that the solution of (4) can be written as a power series in p:

$$V=V_0+p\,V_1+p^2\,V_2+\dots. \tag{5}$$

Setting $p=1$ results in the approximate series solution of (1):

$$u=\lim_{p\to 1}V=V_0+V_1+V_2+\dots. \tag{6}$$

The series in (6) is convergent for most cases, and the rate of convergence depends on the nonlinear operator $A(V)$ [1].

3. Applications

3.1 Modified Boussinesq equation

Firstly, consider the following general equation [9, 10],

$$u_{tt} + \alpha u_{xxt} + \beta u_{xxxx} + \gamma\left(u^n\right)_{xx} = 0, \qquad (7)$$

where α, β, γ and n are real constants.

This equation is called the high-order modified Boussinesq equation with the damping term u_{xxt}. It appears in several domains of mathematics and physics.

When $\alpha = 0,\ \beta = -1,\ \gamma = 1$ and $n = 3$, equation (7) becomes the cubic modified Boussinesq equation which was presented in the famous Fermi–Pasta–Ulam problem and is used to investigate the behavior of systems which are primarily linear but a nonlinearity is introduced as a perturbation. It arises in other physical applications as well.

Example 1.

Consider the cubic modified Boussinesq equation,

$$u_{tt} + u_{xxt} + \frac{2}{9}u_{xxxx} - \left(u^3\right)_{xx} = 0, \quad u(x,0) = 1 + \tanh\left(\frac{3}{2}x\right), \quad u_t(x,0) = -3\,\mathrm{sech}^2\left(\frac{3}{2}x\right). \qquad (8)$$

According to (4), a homotopy $V(x,t,p): \Omega \times [0,1] \to \mathbb{R}$ can be written as

$$(1-p)\left(V_{tt} - u_{0,tt}\right) + p\left(V_{tt} + V_{xxt} + \frac{2}{9}V_{xxxx} - \left(V^3\right)_{xx}\right) = 0, \qquad p \in [0,1],\ (x,t) \in \Omega, \qquad (9)$$

where $u_0 = u_0(x,t) = V_0(x,0) = u(x,0)$ and $u_{0,tt} = \dfrac{\partial^2 u_0}{\partial t^2}$.

One can now try to obtain a solution of (19) in the form of:

$$V(x,t) = V_0(x,t) + p\,V_1(x,t) + p^2\,V_2(x,t) + \ldots. \qquad (10)$$

Substituting (10) into (9) and equating terms of the identical powers of p yields

$$\begin{aligned}
p^0 &: \quad V_{0,tt} = 0,\\
p^1 &: \quad V_{1,tt} + V_{0,xxt} + \frac{2}{9}V_{0,xxxx} - 3\left(V_0\right)^2 V_{0,xx} - 6V_0\left(V_{0,x}\right)^2 = 0,\\
p^2 &: \quad V_{2,tt} + V_{1,xxt} + \frac{2}{9}V_{1,xxxx} - 3\left(V_0\right)^2 V_{1,xx} - 6V_0V_1V_{0,xx} - 6V_1\left(V_{0,x}\right)^2 - 12V_0V_{0,x}V_{1,x} = 0,\\
&\ \ \vdots,
\end{aligned} \qquad (11)$$

with the following initial conditions:

$$V_i(x,0)=\begin{cases}1+\tanh\left(\frac{3}{2}x\right), & i=0,\\ 0, & i=1,2,...,\end{cases} \qquad V_{i,t}(x,0)=\begin{cases}-3\operatorname{sech}^2\left(\frac{3}{2}x\right), & i=0,\\ 0, & i=1,2,...\end{cases}. \tag{12}$$

Solving the system (11), with the conditions (12) gives

$$V_0(x,t)=1+\tanh\left(\frac{3}{2}x\right)-3t\operatorname{sech}^2\left(\frac{3}{2}x\right),$$

$$V_1(x,t)=-9t^2\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^2\left(\frac{3}{2}x\right)-\frac{9}{40}t^3\{80\operatorname{sech}^2\left(\frac{3}{2}x\right)-120\operatorname{sech}^4\left(\frac{3}{2}x\right)$$
$$+120\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^2\left(\frac{3}{2}x\right)-360\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^4\left(\frac{3}{2}x\right)\}-\frac{9}{40}t^4\{450\operatorname{sech}^6\left(\frac{3}{2}x\right)$$
$$-360\operatorname{sech}^4\left(\frac{3}{2}x\right)+675\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^6\left(\frac{3}{2}x\right)-360\tanh\left(\frac{3}{2}x\right)\operatorname{sech}^4\left(\frac{3}{2}x\right)\}$$
$$-\frac{9}{40}t^5\{486\operatorname{sech}^6\left(\frac{3}{2}x\right)-567\operatorname{sech}^8\},$$

$$\vdots. \tag{13}$$

In this manner the other components can be easily obtained. For this example, we continued solving (11) for V_n, n=0,1,… until V_2 and hence obtained the two-term approximation $\varphi_2= V_0+ V_1+ V_2$.

With the aid of Maple software, the rational approximations [N/M] can be obtained by applying PA with respect to t to the obtained series solution φ_2 such that $N+M \leq 9$ (highest power of the variable t in φ_2).

In order to evaluate the reliability and accuracy of the HPM with the enhancement of PA, here we mention that (8) has an exact solution,

$$u(x,t)=1+\tanh\left(\frac{3}{2}x-3t\right). \tag{14}$$

Some numerical results for the absolute error between the exact solution u and series solution φ_2 are presented in Tables 1 and for the absolute error between u and Padé approximation [2/6] are presented in Tables 2.

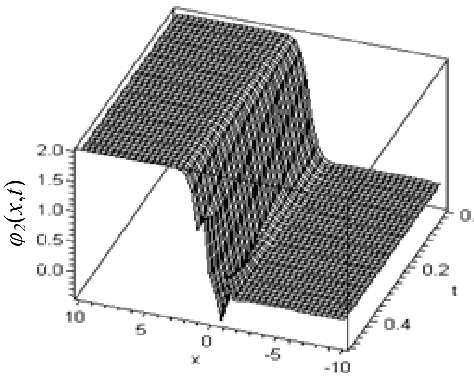
Table 1

Results obtained for the absolute error $|u-\varphi_2|$ for selected values of x and t

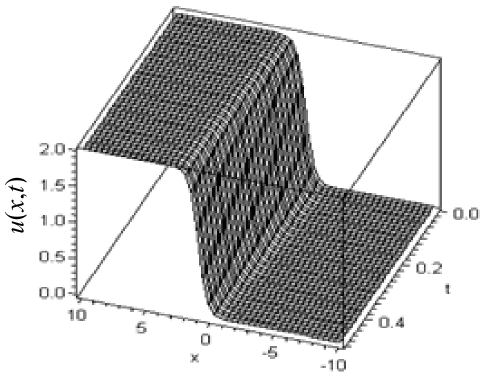
$t_i \backslash x_i$	0.0	0.5	1.0	1.5	2.0
0.1	0.2372496E–3	0.1051763 E–2	0.826206 E–3	0.47619 E–4	0.35346 E–4
0.3	0.6138779 E–2	0.1028504349	0.3866883 E–1	0.2691527 E–2	0.631936 E–3
0.5	0.2827709 E–1	0.2138252261	0.7879245 E–1	0.5840657 E–1	0.8879789 E–1
0.7	3.585856474	3.486293464	0.6280186210	0.3874946876	0.76612254
0.9	43.72139536	29.68321428	4.835834768	1.603806045	1.2539338915
1.1	292.5846898	144.0809202	18.59432219	4.466162969	1.361639355

Table 2
Results obtained for the absolute error $|u-[2/6]|$ between the exact solution and the one obtained by application of a PA [2/6] to HPM series solution φ_2

$t_i \backslash x_i$	0.0	0.5	1.0	1.5	2.0
0.1	0.2434446 E–3	0.1058608 E–2	0.825470 E–3	0.47597 E–4	0.353445 E–4
0.3	0.6522647 E–2	0.5470101 E–1	0.356969 E–1	0.794166 E–3	0.1181480 E–2
0.5	0.4220276 E–2	0.2137442037	0.638468 E–1	0.180639 E–2	0.5222480 E–2
0.7	0.1578839 E–2	0.3489404687	0.145551 E–1	0.670999 E–2	0.2724002 E–1
0.9	0.3238113 E–2	0.6899567854	0.697368 E–1	0.848258 E–1	0.2258921 E–2
1.1	0.2763887E–2	0.4006148959	0.697437 E–1	0.135511 E–1	0.1344292 E–1



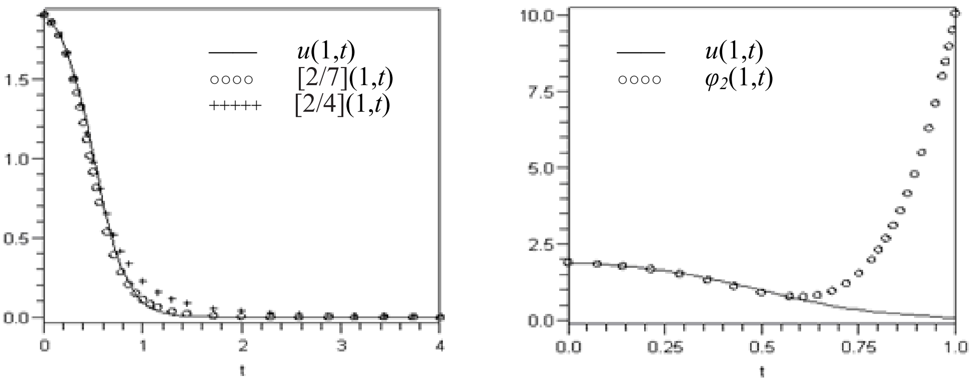
(a)



(b)

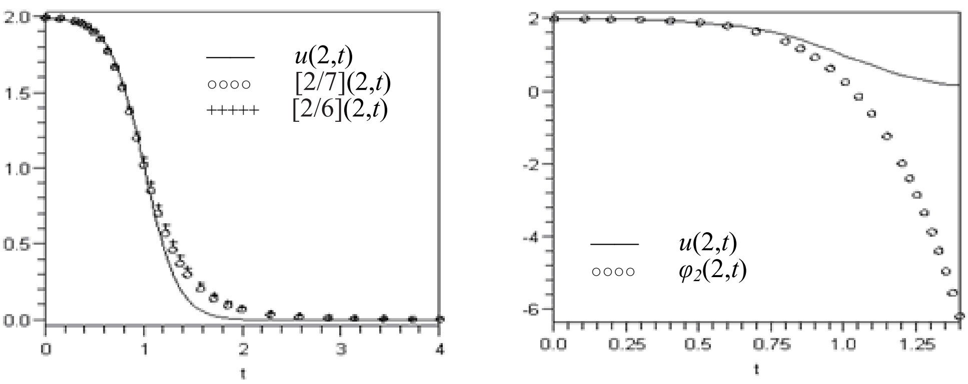
Figure1: The surfaces generated from the HPM result $\varphi_2(x,t)$, shown in (a), in comparison with the exact solution, shown in (b), for the cubic modified Boussinesq equation in the domain $0 \le t \le 0.5$ and $-10 \le x \le 10$

The behavior of the solutions φ_2 and the exact solution illustrated in Figs. 1(a), 1(b), 2(b) and 3(b) shows that the HPM solution φ_2 is an accurate approximation only in the interval $0 \le t \le 0.5$ but diverges for values of $t > 0.5$. We can conclude the same result from the numerical results of the absolute error between the exact solution and φ_2 shown in Table 1 as well. Figs. 2(b) and 3(b) show the deterioration in the HPM solution φ_2 beyond the interval of convergence. So, PA would be needed to increase the convergence domain region.



(a) (b)

Figure 2: Results for x =1, obtained for the exact solution (solid line) and the ones obtained by applying PA [2/4] and [2/7] to φ_2, shown in (a), in comparison with HPM result φ_2, shown in (b), for the cubic modified Boussinesq equation



(a) (b)

Figure 3: Results for x =2, obtained for the exact solution (solid line) and the ones obtained by applying PA [2/6] and [2/7] to φ_2, shown in (a), in comparison with HPM result φ_2, shown in (b), for the cubic modified Boussinesq equation

It's seen in Table 2, Figs. 2(a) and 3(a), that the application of PA, for example [2/4], [2/6] and [2/7], to the truncated series solution φ_2 greatly improves the results of the HPM and increases approximate solution accuracy and convergence domain region.

3.2 Cubic Boussinesq equation

A well-known model of nonlinear dispersive waves which was proposed by Boussinesq is formulated in the form

$$u_{tt} = u_{xx} + 3\left(u^3\right)_{xx} + u_{xxxx}, \qquad L_0 \le x \le L_1 \tag{15}$$

The Boussinesq equation (15) describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice [11].

Example 2.

Consider the cubic Boussinesq equation [12]

$$u_{tt}-u_{xx}-u_{xxxx}+2\left(u^{3}\right)_{xx}=0, \qquad u\left(x,0\right)=\frac{1}{x}, \qquad u_{t}\left(x,0\right)=-\frac{1}{x^{2}}. \tag{16}$$

The homotopy of (16) $V\left(x,t,p\right):\Omega\times[0,1]\to\mathbb{R}$ satisfies

$$\left(1-p\right)\left(V_{tt}-u_{0,tt}\right)+p\left(V_{tt}-V_{xx}-V_{xxxx}+2\left(V^{3}\right)_{xx}\right)=0, \quad p\in[0,1], \ \left(x,t\right)\in\Omega. \tag{17}$$

Suppose the solution of (17) to be in the following form,

$$V\left(x,t\right)=V_{0}\left(x,t\right)+p\,V_{1}\left(x,t\right)+p^{2}\,V_{2}\left(x,t\right)+\dots. \tag{18}$$

Substituting (18) into (17), and equating the terms with the identical powers of p yields

$$\begin{aligned}
p^{0}: &\quad V_{0,tt}=0,\\
p^{1}: &\quad V_{1,tt}-V_{0,xx}-V_{0,xxxx}+6\left(V_{0}\right)^{2}V_{0,xx}+12V_{0}\left(V_{0,x}\right)^{2}=0,\\
p^{2}: &\quad V_{2,tt}-V_{1,xx}-V_{1,xxxx}+6\left(V_{0}\right)^{2}V_{1,xx}+12V_{0}V_{1}V_{0,xx}+12V_{1}\left(V_{0,x}\right)^{2}+24V_{0}V_{0,x}V_{1,x}=0,\\
&\quad \vdots,
\end{aligned} \tag{19}$$

with the following initial conditions:

$$V_{i}\left(x,0\right)=\begin{cases}\dfrac{1}{x}, & i=0,\\ 0, & i=1,2,\dots,\end{cases} \qquad V_{i,t}\left(x,0\right)=\begin{cases}-\dfrac{1}{x^{2}}, & i=0,\\ 0, & i=1,2,\dots\end{cases}. \tag{20}$$

Solving the system (19) corresponding to the initial conditions (20) yields

$$\begin{aligned}
V_{0}\left(x,t\right)&=\frac{1}{x}-\frac{t}{x^{2}},\\
V_{1}\left(x,t\right)&=\frac{t^{2}}{x^{3}}-\frac{t^{3}}{x^{4}}\underbrace{-\frac{15t^{4}}{x^{7}}+\frac{21t^{5}}{5x^{8}}}_{\text{noise terms}},\\
V_{2}\left(x,t\right)&=\frac{t^{4}}{x^{5}}-\frac{t^{5}}{x^{6}}+\underbrace{\frac{15t^{4}}{x^{7}}-\frac{21t^{5}}{5x^{8}}-\frac{308t^{6}}{5x^{9}}-\frac{2250t^{6}}{x^{11}}+\frac{612t^{7}}{35x^{10}}+\frac{1782t^{7}}{7x^{12}}-\frac{11583t^{8}}{35x^{13}}+\frac{273t^{9}}{5x^{14}}}_{\text{noise terms}},\\
&\ \vdots.
\end{aligned} \tag{21}$$

It's noticed that each V_n contains terms that have similar terms in V_{n+1}, n=1,2,… but with a different sign. These terms called by noise terms and will be self-cancelled in $\varphi_{\infty}=\lim_{k\to\infty}\sum_{n=0}^{k}V_{n}$.

In this problem, we continued solving (19) for V_n, n=0,1,… until V_4 and hence obtained the four-term approximation $\varphi_4= V_0+ V_1+ V_2+ V_3+ V_4$,

$$\varphi_4(x,t) = \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{t^3}{x^4} + \frac{t^4}{x^5} - \frac{t^5}{x^6} + \frac{t^6}{x^7} - \frac{t^7}{x^8} + \frac{t^8}{x^9} - \frac{t^9}{x^{10}} + \text{noise terms}. \quad (22)$$

We found that the application of the Padé approximants [*N/M*], with respect to *t* to the series solution φ_4, for all $N \geq 0$ and $M \geq 1$ such that $N+M \leq 9$, gives

$$[N/M](x,t) = \frac{1}{x+t}, \quad (23)$$

which is the exact solution of the cubic Boussinesq equation (16).

It's obvious that utilizing of PA provides self-examination to the exact solution if the NLDE has a rational exact one by using only a few-term solution of the HPM.

4. Conclusions

The first conclusion that can be draw from our results is that the He's homotopy perturbation method is an effective tool to deal with nonlinear dispersive equations of Boussinesq and, in isolation, provides an accurate approximation in a relatively small interval of *t*. The second conclusion is that the application of the Padé approximants to the HPM truncated series solution greatly improves the results, increases accuracy, enlarge convergence region and in some cases gives self-examination to the exact solution. It's worth noting that the HPM with the enhancement of PA is an effective, simple and quite accurate tool for handling and solving nonlinear dispersive equations and other types of NLDEs. The various applications of the HPM prove that it's an efficient method to handle nonlinear structures.

References

[1] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng., 178 (1999), 257–262.

[2] J.H. He, Homotopy perturbation method: a new nonlinear analytical technique, Appl. Math. Comput., 135 (2003), 73–79.

[3] J.H. He, Comparison of homotopy perturbation method and homotopy analysis method, Appl. Math. Comput., 156 (2004), 527–539.

[4] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons Fractals, 26 (2005), 695–700.

[5] T. Ozis and A.Yildirim, A comparative study of He's homotopy perturbation method for determining frequency-amplitude relation of a nonlinear oscillator with discontinuities, Int. J. Nonlinear Sci. Numer. Simul., 8(2) (2007), 243–248.

[6] Q.K. Ghori, M. Ahmed and A.M. Siddiqui, Application of homotopy perturbation method to squeezing flow of a Newtonian fluid, Int. J. Nonlinear Sci. Numer. Simul., 8(2) (2007), 179–184.

[7] M.M. Mousa and S.F. Ragab, Application of the homotopy perturbation method to linear and nonlinear schrödinger equations, Z.Naturforsch. 63a (2008), 140–144.

[8] Baker G.A., Essentials of Padé Approximants, Academic press, New York, 1975.

[9] B. Li, Y. Chen, and H.Q. Zhang, Explicit exact solutions for some nonlinear partial differential equations with nonlinear terms of any order, Czech. J. Phys. 53 (2003), 283–295.

[10] B. Li and Y. Chen, Nonlinear Partial Differential Equations Solved by Projective Riccati Equations Ansatz, Z. Naturforsch., 58a (2003) 511–519.

[11] A.M. Wazwaz, Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method, Chaos, Solitons Fractals, 12 (2001), 1549–1556.

[12] A.M. Wazwaz, The variational iteration method for rational solutions for KdV, K(2,2), Burgers, and cubic Boussinesq equations, J. Comput. Appl. Math. 207(1) (2007), 18–23.

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