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On a Truncated Erlang Queuing System with Bulk Arrivals, Balking and Reneging

M. S. El-Paoumy and M. M. Ismail

Department of Statistics Faculty Of Commerce
Al- Azhar University. Girls Branch, Egypt
mmismail_eg@yahoo.com

Abstract

The aim of this paper is to derive the analytical solution of the queue:M^x /E_k /I/N with balking and reneging in which

i) Units arrive in batches of random size with the interarrival times of batches following negative exponential distribution.

ii) The queue discipline is FCFS, it being assumed that the batches are pre-ordered for service purpose

iii)The service time distribution is Erlangian with K stages.Recurrence relations connecting the various probabilities introduced are found. Some measures of effectiveness as L and L_q are deducted and some special cases are also obtained.

1. Introduction

Many researchers as Cox and Miller [3], Probhu [9] and Saaty [10] studied Markovian queue with bulk arrivals but they did not take consideration of balking and reneging. Some researchers as Burk [2] studied the queue: M^x /M/1 in the homogeneous case with bulking and delays. Other, Cromie et al. [4] discussed the queues: M^x/M/C, Abou-ELAta et al. [1] discussed with the bulk arrival queue M^x/M/1 with both concepts of balking and reneging and the non-truncated queue: M^x/E_k/1 had been solved by Medhi [7] who obtained the solution in terms of generating function.

This paper is aimed to treat the bulk arrival queue for non-Markovian (Erlangian service) queue: M^x/E_k/1/N with balking and reneging. In this work we obtain $P_{n,k}$ (the probabilities that there are n units in the system and the unit in the service occupies $S^{\underline{th}}$ stage (S =1, 2…,k)) in terms of P_0 by using a recurrence relations. Also, we obtain the probability of an empty system P_0, and deduce some special cases.

2. Description of the system

In the present system, it is assumed that the units arrival the system in batches of random size *X*, i.e., at each moment of arrival, there is a probability $C_j = \mathrm{P}(\mathrm{X} = j)$ that, units arrival simultaneously, C_o =0 and the interarrival times of batches follows a negative exponential distribution with time independent parameter λ. Let $\lambda C_j \, \Delta t, (j = 1,2,........,\mathrm{N})$, be the first order probability that batch of *j* units comes in time $\Delta t.$ The service channel is busy if a unit is found in any one of the *K* stages (of M/E_K/1/N). In this case the arrival units form a queue, with capacity N.

Consider the balk concept with probability β= prob.{a unit joins the queue}, where $0 \le \beta < 1$ if *n=1(1) N* and β *=1 if n=0, n* is a number of units in the system.

It is also assumed that the units renege according to an exponential distribution, $f(t) = \alpha e^{-\alpha t}$, $t \ge 0$, $\alpha > 0$, with parameter α. The probability of reneging in a short period of time Δt is a given by:

$$r_n = (n-1)\, \alpha \; \Delta t, \quad for \quad 1 < n \le \mathrm{N} \quad and \quad r_n = 0, \quad for \quad n = 0,1 .$$

Using the rate out = rate in approach the steady–state, difference equations can be written as follows:

For, *n=0,* we have

$$k\mu \mathrm{P}_{i,1} - \lambda \mathrm{P}_0 = 0, \tag{1}$$

For *n=1,* we have,

$$\left.\begin{aligned} &k\mu \mathrm{P}_{1,s+1} - (k\mu + \beta\lambda)\mathrm{P}_{1,s} = 0, && s = 1(1)k-1 \\ &(k\mu + \alpha)\, \mathrm{P}_{2,1} - (k\mu + \beta\lambda)\mathrm{P}_{1,k} + \lambda c_1 \mathrm{P}_0 = 0 && s = k \end{aligned}\right\}$$

(2)

For *n=2(1)N-1,* we have,

$$\left.\begin{aligned} &(k\mu + (n-1)\alpha)\mathrm{P}_{n,s+1} - (k\mu + (n-1)\alpha + \beta\lambda)\mathrm{P}_{n,s} + \beta\lambda \sum_{j=1}^{n-1} C_j P_{n-j,s} = 0 \qquad s = 1(1)k-1 \\ &(k\mu + (n-1)\alpha)\mathrm{P}_{n+1,1} - (k\mu + (n-1)\alpha + \beta\lambda) + \beta\lambda \sum_{j=1}^{n-1} C_j \mathrm{P}_{n-j,k} + \lambda C_n \mathrm{P}_0 = 0, \qquad s = k \end{aligned}\right\} \tag{3}$$

For $n=N$, we have,

$$\left.\begin{array}{ll}(k\mu+(N-1)\alpha)\mathrm{P}_{N,s+1}-(k\mu+(N-1)\alpha+\beta\lambda)\mathrm{P}_{N,s}+\beta\lambda\sum_{j=1}^{n-1}C_jP_{N-j,s}+\beta\lambda\sum_{j=1}^{N}\sum_{i=N-j+1}^{N}C_j\mathrm{P}_{i,s}=0 & s=1(1)k-1\\ \lambda C_NP_o-(k\mu+(N-1)\alpha+\beta\lambda)P_{N,k}+\beta\lambda\sum_{j=1}^{N-1}C_j\mathrm{P}_{N-j,k}+\beta\lambda\sum_{j=1}^{N}\sum_{i=N-j+1}^{N}C_j\mathrm{P}_{i,k}=0, & s=k\end{array}\right\} \quad (4)$$

Summing (2) over s and using (4) we have

$$(k\mu+\alpha)\mathrm{P}_{2,1}=\beta\lambda\sum_{s=1}^{k}\mathrm{P}_{1,s}+(1-C_1)\lambda\,\mathrm{P}_0, \quad (5)$$

also summing (3) over s, using (5) and adding the results obtaining for n=2(1)N-1, we get:

$$\mathrm{P}_{n,1}=\theta_n\left\{\beta\sum_{i=1}^{n-1}\sum_{j=i}^{\mathrm{N}}\sum_{s=1}^{k}C_j\mathrm{P}_{n-j,s}+\sum_{j=n}^{\mathrm{N}}C_j\mathrm{P}_0\right\}, \quad (6)$$

where:

$$\theta_n=\frac{\lambda}{k\mu+(n-1)\alpha}. \qquad n=1(1)\mathrm{N}$$

From the 1st equation of (2): we have

$$\mathrm{P}_{1,s}=\theta_1(1+\beta\theta_1)^{s-1}\mathrm{P}_0. \qquad s=1(1)k \quad (7)$$

Thus upon using 1st equations of (3) and (6), we get for *n=2(1) N-1*

$$\left.\begin{array}{ll}P_{n,1}=\theta_n\left(\beta\sum_{i=1}^{n-1}\sum_{j=i}^{\mathrm{N}}\sum_{s=1}^{k}C_j\mathrm{P}_{n-j,s}+\sum_{j=n}^{\mathrm{N}}C_j\mathrm{P}_0\right) & \\ \mathrm{P}_{n,s+1}=(1+\beta\theta_n)\mathrm{P}_{n,s}-\beta\theta_n\sum_{j=1}^{n-1}C_j\mathrm{P}_{n-j,s} & s=1(1)k-1\end{array}\right\}. \quad (8)$$

Also, from 1st equation of (4) and equation (6) at $n=N$

$$\left.\begin{array}{l}P_{N,1}=\theta_N\left(\beta\sum_{i=1}^{\mathrm{N}-1}\sum_{j=i}^{\mathrm{N}}\sum_{s=1}^{k}C_j\mathrm{P}_{\mathrm{N}-i,s}+C_j\mathrm{P}_{\mathrm{N}-i,s}+C_\mathrm{N}\mathrm{P}_0\right)\\ P_{N,s+1}=(1+\beta\theta_\mathrm{N})\mathrm{P}_{\mathrm{N},s}-\beta\theta_\mathrm{N}\sum_{j=1}^{\mathrm{N}-1}C_j\mathrm{P}_{\mathrm{N}-j,s}-\beta\theta_\mathrm{N}\sum_{j=1}^{\mathrm{N}}\sum_{i=\mathrm{N}-j+1}^{\mathrm{N}}C_j\mathrm{P}_{i,s}\end{array}\right\}. \quad (9)$$

Equations (7), (8) and (9) are required recurrence relations, which give all probabilities in terms of P_0 which itself may now be determined by using the normalizing condition $\mathrm{P}_o+\sum_{n=1}^{\mathrm{N}}\sum_{s=1}^{k}\mathrm{P}_{n,s}=1$, hence all the probabilities are completely in terms of the queue parameters.

The following example illustrates the discussed method.

3. Example

In the above system: $M^x/E_k/I/N(\alpha\beta)$ with balking and reneging, letting k=2, N=3, i.e. the queue: $M^x/E_2/I/4\ (\alpha\beta)$ with balking and reneging, the results are:

$$P_{1,1} = a_1 P \qquad P_{1,2} = a_2 P_0,$$
$$P_{2,1} = b_1 P \qquad P_{2,2} = b_2 P_0,$$
$$P_{3,1} = g_1 P \qquad P_{3,2} = g_2 P_0,$$
$$P_{4,1} = f_1 P \qquad P_{4,2} = g_2 P_0,$$

where:

$$\theta_n = \frac{\lambda}{2\mu + (n-1)\alpha}, \qquad n = 1(1)N, \qquad \theta_1 = \frac{\lambda}{2\mu},$$

$$a_1 = \theta, \qquad a_2 = \theta_1(1+\beta\theta_1),$$

$$b_1 = \theta_2\{\beta(a_1+a_2) + (C_2+C_3+C_4)\},$$

$$b_2 = (1+\beta\theta_2)b_1 - \beta\theta_2 C_1 a_1,$$

$$g_1 = \theta_3\{\beta(b_1+b_2) + \beta(C_2+C_3+C_4)(a_1+a_2) + C_3 + C_4\},$$

$$g_2 = (1+\beta\theta_3)g_1 - \beta\theta_3(C_1 b_1 + C_2 a_1),$$

$$f_1 = \theta_4\{\beta(g_1+g_2) + \beta(C_2+C_3+C_4)(b_1+b_2) + \beta(C_3+C_4)(a_1+a_2) + C_4\},$$

$$f_2 = (1+\beta\theta_4)f_1 - \beta\theta_4\{C_1(g_1+f_1) + C_2(b_1+g_1+f_1) + (a_1+b_1+g_1+f_1)(C_3+C_4)\}$$

form the normalizing condition:

$$P_0 = \{1 + a_1 + a_2 + b_1 + b_2 + g_1 + g_2 + f_1 + f_2\}^{-1}.$$

Therefore, the expected number in the system and in the queue is respectively.

$$L = \sum_{n=1}^{N}\sum_{s=1}^{k} nP_{n,s},$$
$$L = \{(a_1+a_2) + 2(b_1+b_2) + 3(g_1+g_2) + 4(f_1+f_2)\}P_0,$$
$$L_q = \sum_{n=1}^{N}\sum_{s=1}^{k}(n-1)P_{n,s},$$
$$L_q = \{(b_1+b_2) + 2(g_1+g_2) + 3(f_1+f_2)\}P_0.$$

4. Special cases:

1- Let k=1, in Equations (7), (8) and (9), then we get the bulk queue:

$M^X/M/1/N \left(\alpha\ ,\rho\right)$ and the results are:

$$P_1 = \rho P_0\ , \qquad P_2 = \frac{\rho\left(\beta\rho + 1 - c_1\right)}{\delta + 1} P_0\ ,$$

$$P_n = \frac{\beta\rho}{(n-1)\delta + 1} \sum_{j=1}^{n-1} c_j\, P_{n-j} - \frac{\eta_n}{(n-1)\delta + 1}, \qquad n = 3(1)N - 1,$$

$$P_N = \frac{\rho}{\beta\rho + (N-1)\delta + 1} \left[\beta \sum_{j=1}^{N-1} c_j P_{N-j} + \beta \sum_{i=1}^{N} \sum_{j=N-j+1}^{N} c_i P_j + c_N P_0 \right],$$

which agree with the results of El-Paoumy, M.S. [5], where $\rho = \dfrac{\lambda}{\mu}$, $\delta = \dfrac{\alpha}{\mu}$,

$$\eta_n = \beta\rho \sum_{i=1}^{n-1} \sum_{j=q}^{n-i} c_i\, p_j - \beta\rho \sum_{i=1}^{n=1} p_i - \rho p_0 \sum_{i=n}^{N} c_i, \qquad n = 3(1)N - 1$$

2- If we put $c_j = \delta_{j1}$, we get the system: $M/E_k/1/N$ with balking and reneging

3- Let $K = 1$, $c_j = \delta_{j1}$, we get the system: M/M/1/N $\left(\alpha, \beta\right)$

$$P_1 = \rho P_0\ , \qquad P_2 = \frac{\rho\left(\beta\rho\right)}{\delta + 1} P_0$$

$$P_n = \frac{\beta\rho}{(n-1)\delta + 1} P_{n-1}, \qquad n = 3(1)N\ .$$

Or $$P_n = \frac{\rho(\beta\gamma)^{n-1}}{(\delta'+1)_{n-1}} P_0, \qquad n = 1(1)N,$$

$$P_0 = \left[1 + \rho + \rho \sum_{n=2}^{N} \frac{(\beta\gamma)^{n-1}}{(\delta'+1)_{n-1}} \right]^{-1},$$

where $\gamma = \dfrac{\lambda}{\alpha}$, $\delta' = \dfrac{\mu}{\alpha}$, $(\delta')_n = \delta'(\delta'+1)\ldots(\delta'+n-1),\ n \geq 1\ and\ (\delta')_0 = 1.$

4- Moreover, let $\beta = 1$ and $\alpha = 0$, we have the system; M/M/1/N without bulking or reneging which were studied by White et al. [11], Gross and Herris [6], Morse [8] and others.

The effects of λ, β and α on L and L_q are showing by tables (1) to (5) , for μ=6, C_1 =0.4, C_2 =0.3, C_3 =0.1, C_4 =0.2.

Table (1):- α=0.1

β	λ	1	2	3	4	5	10	20	30	50
0.1	L	0.48540	0.79099	1.00435	1.16410	1.28992	1.67983	2.08003	2.33679	2.68353
	L_q	0.22590	0.37289	0.47939	0.56235	0.63038	0.86480	1.16015	1.37928	1.69895
0.2	L	0.50002	0.83254	1.07521	1.26395	1.41762	1.92574	2.46952	2.77714	3.07759
	L_q	0.23572	0.40218	0.53136	0.63800	0.72981	1.07409	1.51797	1.79697	2.08244
0.3	L	0.51502	0.87553	1.14851	1.36656	1.54741	2.15376	2.74988	3.01837	3.16155
	L_q	0.24583	0.43267	0.58557	0.71656	0.83211	1.27125	1.77933	2.02830	2.16359
0.4	L	0.53040	0.91988	1.22378	1.47071	1.67697	2.35539	2.93704	3.13188	2.99148
	L_q	0.25622	0.46432	0.64170	0.79708	0.93534	1.44782	1.95549	2.13738	1.99270
0.5	L	0.54615	0.96549	1.30056	1.57522	1.80420	2.52762	3.05536	3.15724	1.80983
	L_q	0.26690	0.49705	0.69937	0.87859	1.03769	1.60025	2.06740	2.16065	0.81124
0.6	L	0.56227	1.01226	1.37836	1.67899	1.92733	2.67121	3.12329	3.09736	6.40944
	L_q	0.27786	0.53078	0.75821	0.96015	1.13760	1.72843	2.13152	2.09974	5.40855
0.7	L	0.57876	1.06008	1.45670	1.78100	2.04497	2.78883	3.15141	2.90065	4.61685
	L_q	0.28911	0.56545	0.81784	1.04093	1.23380	1.83417	2.15732	1.90259	3.61672
0.8	L	0.59561	1.10882	1.53511	1.88040	2.15609	2.88386	3.14306	2.30474	4.31657
	L_q	0.30064	0.60095	0.87788	1.12016	1.32532	1.92008	2.14753	1.30680	3.31652
0.9	L	0.61283	1.15836	1.61314	1.97645	2.26005	2.95961	3.09367	2.73902	4.19897
	L_q	0.31245	0.63719	0.93796	1.19720	1.41149	1.98881	2.09724	3.73329	3.19895
1.0	L	0.63040	1.20858	1.69038	2.06858	2.35649	3.01902	2.98600	6.41320	4.13849
	L_q	0.32454	0.67409	0.99774	1.27152	1.49190	2.04281	1.98905	5.41173	3.13849

Table (2):- α = 0.3

β \ λ		1	2	3	4	5	10	20	30	50
0.1	L	0.46504	0.76241	0.97191	1.12962	1.25422	1.64118	2.03682	2.29038	2.63703
	L_q	0.21434	0.35583	0.45914	0.53994	0.60633	0.83508	1.12242	1.33650	1.65421
0.2	L	0.47848	0.80097	1.03813	1.22340	1.37464	1.87623	2.41733	2.73010	3.04973
	L_q	0.22331	0.38285	0.50739	0.61052	0.69946	1.03392	1.47053	1.75239	2.05532
0.3	L	0.49224	0.84084	1.10666	1.31995	1.49745	2.09697	2.69999	2.98380	3.16777
	L_q	0.23254	0.41096	0.55774	0.68394	0.79559	1.22364	1.73307	1.99520	2.17012
0.4	L	0.50634	0.88198	1.17713	1.41826	1.62067	2.29521	2.89533	3.11446	3.08469
	L_q	0.24202	0.44013	0.60994	0.75942	0.89309	1.39624	1.91638	2.12082	2.08600
0.5	L	0.52077	0.92430	1.24916	1.51733	1.74246	2.46745	3.02435	3.16283	2.56966
	L_q	0.25175	0.47031	0.66368	0.83614	0.99038	1.54783	2.03822	2.16673	1.57077
0.6	L	0.53553	0.96772	1.32234	1.61619	1.86120	2.61355	3.10458	3.14166	5.528867
	L_q	0.26174	0.50143	0.71865	0.91331	1.08606	1.67758	2.11409	2.14432	4.626228
0.7	L	0.55062	1.01215	1.39627	1.71394	1.97559	2.73537	3.14737	3.03313	5.04925
	L_q	0.27198	0.53343	0.77453	0.99016	1.17896	1.78659	2.15417	2.03515	4.04897
0.8	L	0.56604	1.05749	1.47054	1.80979	2.08459	2.83564	3.15827	2.74942	4.45282
	L_q	0.28247	0.56624	0.83100	1.06603	1.26812	1.87687	2.16335	1.75124	3.45274
0.9	L	0.58179	1.10365	1.54477	1.90304	2.18750	2.91728	3.13750	1.82858	4.26688
	L_q	0.29321	0.59980	0.88775	1.14031	1.35285	1.95072	2.14147	0.83081	3.26685
1.0	L	0.59786	1.15053	1.61857	1.99315	2.28386	2.98297	3.07864	4.33447	4.17963
	L_q	0.30421	0.63401	0.94448	1.21249	1.43267	2.01033	2.08191	5.29123	3.17961

Table (3):- α=0.5

β \ λ		1	2	3	4	5	10	20	30	50
0.1	L	0.44656	0.73625	0.94209	1.09785	1.22130	1.60556	1.99704	2.24738	2.59290
	L_q	0.20400	0.34047	0.44085	0.51965	0.58453	0.80812	1.08803	1.29712	1.61188
0.2	L	0.45895	0.77215	1.00415	1.18616	1.33511	1.83050	2.36824	2.68464	3.02010
	L_q	0.21223	0.36549	0.48579	0.58570	0.67200	0.99729	1.42623	1.70950	2.02650
0.3	L	0.47163	0.80926	1.06840	1.27722	1.45153	2.04404	2.65168	2.94828	3.16477
	L_q	0.22070	0.39151	0.53271	0.65450	0.76253	1.17976	1.68851	1.96127	2.16746
0.4	L	0.48461	0.84754	1.13454	1.37018	1.56882	2.23844	2.85345	3.09305	3.13571
	L_q	0.22938	0.41850	0.58139	0.72540	0.85473	1.34806	1.87726	2.10036	2.13715
0.5	L	0.49789	0.88693	1.20225	1.46417	1.68539	2.40986	2.99137	3.15910	2.85382
	L_q	0.23830	0.44641	0.63158	0.79771	0.94722	1.49810	2.00723	2.16358	1.85488
0.6	L	0.51147	0.92735	1.27120	1.55836	1.79978	2.55750	3.08178	3.16402	1.03805
	L_q	0.24744	0.47522	0.68303	0.87074	1.03878	1.62854	2.09271	2.16701	0.03964
0.7	L	0.52534	0.96875	1.34103	1.65196	1.91075	2.68252	3.13613	3.10441	6.23598
	L_q	0.25680	0.50485	0.73547	0.94383	1.12830	1.73987	2.14395	2.10661	5.23525
0.8	L	0.53951	1.01103	1.41140	1.74423	2.01731	2.78708	3.16112	2.94451	4.67554
	L_q	0.26640	0.53526	0.78862	1.01638	1.21489	1.83360	2.16693	1.94632	3.67540
0.9	L	0.55397	1.05413	1.48197	1.83455	2.11871	2.87365	3.15929	2.54598	4.36236
	L_q	0.27621	0.56640	0.84223	1.08783	1.29783	1.91162	2.16377	1.54776	3.36231
1.0	L	0.56873	1.09796	1.55242	1.92235	2.21445	2.94468	3.12892	1.03696	4.23343
	L_q	0.28626	0.59819	0.89601	1.15770	1.37662	1.97590	2.13253	0.03960	3.23341

Table (4):- α = 0.8

B	λ	1	2	3	4	5	10	20	30	50
0.1	L	0.42176	0.70081	0.90148	1.05446	1.17627	1.55689	1.94285	2.18848	2.53098
	L_q	0.19036	0.32005	0.41642	0.49252	0.55534	0.77198	1.04177	1.24363	1.55275
0.2	L	0.43280	0.73325	0.95807	1.13553	1.28129	1.76795	2.29994	2.61963	2.97399
	L_q	0.19766	0.34249	0.45707	0.55261	0.63533	0.94799	1.36513	1.64847	1.98172
0.3	L	0.44410	0.76676	1.01668	1.21926	1.38907	1.97106	2.58239	2.89444	3.14976
	L_q	0.20515	0.36580	0.49950	0.61531	0.71837	1.12008	1.62504	1.90999	2.15305
0.4	L	0.45565	0.80132	1.07707	1.30499	1.49822	2.15917	2.79119	3.05632	3.17272
	L_q	0.21283	0.38997	0.54355	0.68009	0.80336	1.28158	1.81942	2.06523	2.17444
0.5	L	0.46746	0.83687	1.13901	1.39203	1.60742	2.32823	2.93990	3.14286	3.04244
	L_q	0.22070	0.41496	0.58905	0.74642	0.88919	1.42836	1.95900	2.14833	2.04356
0.6	L	0.47952	0.87337	1.20224	1.47970	1.71544	2.47677	3.04277	3.17471	2.51833
	L_q	0.22876	0.44075	0.63580	0.81375	0.97482	1.55857	2.05607	2.17832	1.51935
0.7	L	0.49183	0.91077	1.26647	1.56735	1.82117	2.60508	3.11085	3.15788	6.03577
	L_q	0.23702	0.46730	0.68359	0.88154	1.05930	1.67200	2.12040	2.16045	7.03148
0.8	L	0.50440	0.94902	1.33145	1.65436	1.92369	2.71458	3.15153	3.08306	5.46973
	L_q	0.24547	0.49457	0.73222	0.94929	1.14181	1.76948	2.15860	2.08505	4.46933
0.9	L	0.51723	0.98805	1.39689	1.74015	2.02226	2.80713	3.16899	2.91257	4.60106
	L_q	0.25411	0.52253	0.78146	1.01651	1.22168	1.85238	2.17440	1.91428	3.60095
1.0	L	0.53030	1.02779	1.46253	1.82422	2.11632	2.88478	3.16458	2.51661	4.35100
	L_q	0.26295	0.55111	0.83110	1.08277	1.29836	1.92227	2.16886	1.51831	3.35095

Table (5):- α =1.0

β	λ	1	2	3	4	5	10	20	30	50
0.1	L	0.40687	0.67934	0.87675	1.02796	1.14873	1.52714	1.90984	2.15247	2.49238
	L_q	0.18230	0.30790	0.40183	0.47627	0.53785	0.75031	1.01396	1.21121	1.51605
0.2	L	0.41714	0.70977	0.93013	1.10475	1.24852	1.72974	2.25762	2.57843	2.94282
	L_q	0.18906	0.32885	0.43997	0.53288	0.61342	0.91834	1.32763	1.61002	1.95151
0.3	L	0.42764	0.74118	0.98541	1.18411	1.35110	1.92619	2.53835	2.85869	3.13528
	L_q	0.19600	0.35059	0.47977	0.59196	0.69199	1.08388	1.58499	1.87608	2.13903
0.4	L	0.43838	0.77356	1.04240	1.26550	1.45528	2.10991	2.75045	3.02993	3.18263
	L_q	0.20311	0.37313	0.52112	0.65310	0.77263	1.24076	1.78177	2.04001	2.18456
0.5	L	0.44934	0.80688	1.10090	1.34831	1.55989	2.27686	2.90496	3.12748	3.10584
	L_q	0.21039	0.39643	0.56385	0.71583	0.85435	1.38495	1.92640	2.13370	2.10705
0.6	L	0.46053	0.84108	1.16069	1.43195	1.66382	2.42525	3.01474	3.17265	2.79869
	L_q	0.21784	0.42047	0.60781	0.77969	0.93624	1.51436	2.02978	2.17675	1.79966
0.7	L	0.47195	0.87614	1.22154	1.51586	1.76607	2.55494	3.09035	3.17440	1.25262
	L_q	0.22547	0.44523	0.65282	0.84419	1.01744	1.62844	2.10119	2.17728	0.25401
0.8	L	0.48361	0.91200	1.28322	1.59946	1.86577	2.66690	3.13926	3.13052	7.49281
	L_q	0.23327	0.47067	0.69870	0.90889	1.09719	1.72765	2.14729	2.13269	6.49167
0.9	L	0.49549	0.94862	1.34548	1.68223	1.96219	2.76266	3.16617	3.02307	4.89508
	L_q	0.24125	0.49675	0.74527	0.97335	1.17484	1.81305	2.17230	2.02485	3.89488
1.0	L	0.50761	0.98595	1.40809	1.76372	2.05476	2.84397	3.17344	2.79688	4.47002
	L_q	0.24940	0.52345	0.79234	1.03718	1.24986	1.88596	2.17825	2.79850	3.46995

5. Conclusion

In this paper, we derive an exact all recurrence relations to study the steady-state probability that there are n units in the system and the unit in the service being in the s^{th} stage. We also derive analytical expressions for steady- state system performance measures, as L and Lq in terms of P_o which can be computed using the normalizing condition. Some numerical results are given in Tables (1) - (5) for special cases. From the numerical results, it is clear that L and L_q are increasing as λ and β are increasing.

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