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Multivalent Meromorphic Starlike Functions with Negative Coefficients

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Abstract

Coefficient inequalities, distortion theorems and class preserving integral operators are obtained for p-valent meromorphic starlike functions of order α $(0 \leq \alpha < p)$ with negative coefficients.

Keywords: Meromorphic, Starlike, Negative Coefficients

1. Introduction

Let Σ_p denote the class of the form

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k z^k (a_{-p} \neq 0;\ p \in N = \{1, 2, \cdots\}) \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Define

$$D^0 f(z) = f(z) \tag{1.2}$$

$$D^0 f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (p + k + 1) a_k z^k = \frac{(z^{p+1} f(z))}{z^p} \tag{1.3}$$

$$D^2 f(z) = D^1 f(z)) \tag{1.4}$$

and for $n = 1, 2, \cdots$.

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (p+k+1)^n a_k z^k = \frac{(z^{p+1} D^{n-1} f(z))}{z^p}. \quad (1.5)$$

In [1] the authers obtained new criteria for meromorphic p-valent starlike functions of order α $(0 \le \alpha < p)$ via the basic inclusion relationship $B_{n+1}(\alpha) \subset B_n(\alpha) \subset B_n(\alpha), 0 \le \alpha < p, n \in N_0 = N \smile \{0\}$ where $B_n(\alpha)$ is the class consisting of functions in Σ_p satisfying

$$Re \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - (p+1) \right\} < -\alpha (z \in U^*, 0 \le \alpha < p, n \in N_0). \quad (1.6)$$

We note that $B(\alpha) = \Sigma_p^*(\alpha)$ (the class of p-valent meromorphic starlike functions of order α).

Let σ_p be the subclass of Σ_p which consisting of functions of the form

$$f(z) = \frac{a_{-p}}{z^p} - \sum_{k=1}^{\infty} a_k z^k (a_{-p} > 0; a_k \ge 0; p \in N). \quad (1.7)$$

Further let

$$\sigma_p(n, \alpha, \beta) = B_n(\alpha, \beta) \cap \sigma_p. \quad (1.8)$$

Definition : Let $f(z) \in \Sigma_p^*(\alpha)$ be defined by (1.7). Then $f(z) \in \sigma_p(n, \alpha, \beta)$ if and only if

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{2\beta \left[\frac{D^{n+1} f(z)}{D^n f(z)} - p \right] - \left[\frac{D^{n+1} f(z)}{D^n f(z)} + 1 - 2\alpha \right]} \right| < 1$$

for $|z| < 1, 0 \le \alpha < p, \frac{1}{2} < \beta \le 1$.

In the present paper coefficient inequalities, distortion theorem and closure theorms for the class $\sigma_p(n, \alpha, \beta)$ are obtainted. Techniques used are similar to those Silverman [2]. Finally, the class preserving integrators of form

$$f_{c+p-1}(z) = c \int_0^1 u^{c+p-1} f(uz) du \quad (0 \le u < 1, 0 < c < \infty) \quad (1.9)$$

is considered.

2. Coefficient Inequalities

Throrem 1 : Let the function $f(z)$ be defined by (1.7). If

$$\sum_{k=1}^{\infty} (p+k+1)^n [\beta(k+1) - (1-\alpha)] |a_k| \le (\beta(p-1) + 1 - \alpha) |a_{-p}| \quad (2.1)$$

then $f(z) \in B_n(\alpha, \beta)$.

Proof : It suffuces to show that

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{2\beta\left[\frac{D^{n+1}f(z)}{D^n f(z)} - p\right] - \left[\frac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right| < 1, |z| < 1. \tag{2.2}$$

We have

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{2\beta\left[\frac{D^{n+1}f(z)}{D^n f(z)} - p\right] - \left[\frac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right|$$
$$= \left| \frac{\sum\limits_{k=1}^{\infty}(p+k+1)^n(p+k)a_k z^{p+k}}{-2(\beta(p-1)+1-\alpha)|a_{-p}| - \sum\limits_{k=1}^{\infty}(p+k+1)^n[(k+1)(2\beta+1)(2\beta-1)-p-1+2\alpha]a+kz^{p+k}} \right|$$
$$\leq \frac{\sum\limits_{k=1}^{\infty}(p+k+1)^n(p+k)|a_k|}{2(\beta(p-1)+1-\alpha)|a_{-p}| - \sum\limits_{k=1}^{\infty}(p+k+1)^n[(k+1)(2\beta-1)-p-1+2\alpha]|a_k|}.$$

The last expression is bounded by 1 if

$$\sum_{k=1}^{\infty}(p+k+1)^n(p+k)|a_k| \leq 2(\beta(p-1)+1-\alpha)|a_{-p}|$$
$$-\sum_{k=1}^{\infty}(p+k+1)^n[(k+1)(2\beta-1)-p-1+2\alpha]|a_k|$$

which reduces to

$$\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]|a_k| \leq (\beta(p-1)+1-\alpha)|a_{-p}|. \tag{2.3}$$

But (2.3) is true by hypothesis. Hence the result follows.
Theorem 2 : Let the function $f(z)$ be defined by (1.7) then $f(z) \in \sigma_p(n, \alpha, \beta)$ if and only if

$$\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]|a_k| \leq (\beta(p-1)+1-\alpha)|a_{-p}|. \tag{2.4}$$

Proof : In view of Theorem 1, it is sufficient to prove the " only if" part. Let us assume that $f(z)$ defined by (1.7) is in $\sigma_p(n, \alpha, \beta)$. Then

$$\left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)} - 1}{2\beta\left[\frac{D^{n+1}f(z)}{D^n f(z)} - p\right] - \left[\frac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha\right]} \right| < 1,$$

reduces to

$$\left|\frac{\sum_{k=1}^{\infty}(p+k+1)^n(p+k)a_k z^{p+k}}{-2(\beta(p-1)+1-\alpha)|a_{-p}|-\sum_{k=1}^{\infty}(p+k+1)^n[(k+1)(2\beta+1)(2\beta-1)-p-1+2\alpha]a+kz^{p+k}}\right|<1.$$

Hence

$$\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]|a_k| \leq (\beta(p-1)+1-\alpha)|a_{-p}|.$$

Thus the result follows.

Corollary 1 : Let the function $f(z)$ defined by (1.7) be in the class $\sigma_p(n,\alpha,\beta)$. Then

$$a_k \leq \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}. \tag{2.6}$$

Then result is sharp for the function

$$f(z)=\frac{a_{-p}}{z^p}-\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}\quad z^k \quad (k \geq 1). \tag{2.7}$$

Distortion Theorem

Theorem 3 : Let the function $f(z)$ defined by (1.7) be in the class $\sigma_p(n,\alpha,\beta)$. Then $0<|z|=r<1$

$$\frac{a_{-p}}{r^p}-\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}r \leq |f(z)| \leq \frac{a_{-p}}{r^p}+\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}r, \tag{3.1}$$

where equality holds for the function

$$f(z)=\frac{a_{-p}}{z^p}-\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta--1+\alpha)}\quad z\ (z=ir,r), \tag{3.2}$$

and

$$\frac{pa_{-p}}{r^{p-1}}-\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)} \leq |f'(z)| \leq \frac{a_{-p}}{r^{p-1}}+\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)} \tag{3.3}$$

where equality holds for the function $f(z)$ given by (3.3) at $z=\mp ir,\mp r$.

Proof : In view of Theorem 2, we have

$$\sum_{k=1}^{\infty}a_k \leq \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}. \tag{3.4}$$

Thus, for $0<|z|=r<1$,

$$|f(z| \leq \frac{a_{-p}}{r^{p-1}}+r\sum_{k=1}^{\infty}a_k \leq \frac{a_{-p}}{r^{p-1}}+\frac{\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}\ r \tag{3.5}$$

and

$$|f(z)| \geq \frac{a_{-p}}{r^p} - r\sum_{k=1}^{\infty} a_k \geq \frac{a_{-p}}{r^{p-1}} + \frac{\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}\, r. \tag{3.6}$$

Thus (3.1) follows.

Since

$$(p+2)^n[(n+1)\beta-1+\alpha]\sum_{k=1}^{\infty} k|a_k| \leq \sum_{k=1}^{\infty}(p+k+1)^n[(k+1)\beta-1+\alpha]|a_k|$$
$$\leq (\beta(p-1)+1-\alpha)a_{-p}$$

where

$$\frac{(p+k+1)^n[(k+1)\beta-1+\alpha}{k}$$

is an increasing function of k, from Theorem 2 it follows that

$$\sum_{k=1}^{\infty} ka_k \leq \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}. \tag{3.7}$$

Hence,

$$\begin{aligned}
|f'(z)| &\leq \frac{pa_{-p}}{r^{p-1}} + \sum_{k=1}^{\infty} k|a_k|r^{k-1} \\
&\leq \frac{pa_{-p}}{r^{p-1}} + \sum_{k=1}^{\infty} k|a_k| \\
&\leq \frac{pa_{-p}}{r^{p-1}} + \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}
\end{aligned}$$

and

$$\begin{aligned}
|f'(z)| &\geq \frac{pa_{-p}}{r^{p-1}} - \sum_{k=1}^{\infty} k|a_k|r^{k-1} \\
&\geq \frac{pa_{-p}}{r^{p-1}} - \sum_{k=1}^{\infty} k|a_k| \\
&\geq \frac{pa_{-p}}{r^{p-1}} - \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}
\end{aligned}$$

Thus (3.3) follows. It can be easily seen that the function $f(z)$ defined by (3.2) is extremal for the theorem.

4. Closure Theorems

Let the functions $f_j(z)$ be defined for $j = 1, 2, \cdots, m$, by

$$f_j(z) = \frac{a_{-p,j}}{z^p} - \sum_{k=1}^{\infty} a_{k,j} z^k (a_{-p,j} > 0; \quad a_{k,j} \geq 0) \tag{4.1}$$

for $z \in U^*$.

We shall prove the following closure theorems for the class $\sigma_p(n, \alpha, \beta)$.

Theorem 4 : Let the functions $f(z)$ defined by (4.1) be in the class $\sigma_p(n, \alpha, \beta)$ for every $j = 1, 2, \cdots, m$. Then the function $F(z)$ defined by

$$F(z) = \frac{b_{-p}}{z^p} - \sum_{k=1}^{\infty} b_k z^k (b_{-p} > 0; b_k \geq 0; p \in N) \tag{4.2}$$

is a memebr of the class $\sigma_p(n, \alpha, \beta)$, where

$$b_{-p} = \frac{1}{m} \sum_{j=1}^{m} a_{-p,j} \quad \text{and} \quad b_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j} \quad (k = 1, 2, \cdots). \tag{4.3}$$

Proof : Since $f_j(z) \in \sigma_p(n, \alpha, \beta)$ it follows from Theorem 2 that

$$\sum_{k=1}^{\infty} (p + k + 1)^n [\beta(k + 1) - (1 - \alpha)] |a_k| \leq (\beta(p - 1) + 1 - \alpha) |a_{-p}| \tag{4.4}$$

for every $j = 1, 2, \cdots, m$. Hence,

$$\begin{aligned}
&\sum_{k=1}^{\infty} (p + k + 1)^n [\beta(k + 1) - (1 - \alpha)] b_k \\
&= \sum_{k=1}^{\infty} (p + k + 1)^n [\beta(k + 1) - (1 - \alpha)] \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) \\
&= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} (p + k + 1)^n [\beta(k + 1) - (1 - \alpha)] a_{k,j} \right) \\
&\leq (\beta(p - 1) + 1 - \alpha) \left(\frac{1}{m} \sum_{j=1}^{m} a_{-p,j} \right) = (\beta(p - 1) + 1 - \alpha) b_{-p}
\end{aligned}$$

which (in view of Theorem 2) implies that $Fz) \in \sigma_p(n, \alpha, \beta)$.

Theorem 5 : The class $\sigma_p(n, \alpha, \beta)$ is closed under convex linear combination.

Proof : Let the function $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $\sigma_p(n, \alpha, \beta)$, it is sufficient to prove that the function

$$H(z) = \lambda f_1(z) + (1 - \lambda) f_2(z) \quad (0 \leq \lambda \leq 1)$$

is also in the class $\sigma_p(n,\alpha,\beta)$. Since, for $0 \le \lambda \le 1$,

$$H(z) = \frac{\lambda a_{-p,1} + (1-\lambda)a_{-p,2}}{z^p} - \sum_{k=1}^{\infty}\{\lambda a_{k,1} + (1-\lambda)a_{k,2}\}z^k \tag{4.6}$$

we observe that

$$\begin{aligned}
&\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]\{\lambda a_{k,1}+(1-\lambda)a_{k,2}\}\\
&= \lambda\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]a_{k,1}\\
&+(1-\lambda)\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]a_{k,2}\\
&\le (\beta(p-1)+1-\alpha)
\end{aligned} \tag{4.7}$$

with the aid of Theorem 2. Hence $H(z) \in \sigma_p(n,\alpha,\beta)$. This completes the proof of Theorem 5.

Theorem 6 : Let

$$f_0(z) = \frac{a_{-p}}{z^p} \tag{4.8}$$

and

$$f_k(z) = \frac{a_{-p}}{z^p} - \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}\ z^k \quad (k \ge 1). \tag{4.9}$$

Then $f(z) \in \sigma_p(n,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty}\lambda_k f_k(z), \tag{4.10}$$

where

$$\lambda_k \ge 0` \ (k \ge 0) \ \text{ and } \ \sum_{k=0}^{\infty}\lambda_k = 1.$$

Proof : Let

$$f(z) = \sum_{k=0}^{\infty}\lambda_k f_k(z) \ \text{ where } \ \lambda_k \ge 0 \ \ (k \ge 0) \ \text{ and } \ \sum_{k=1}^{\infty}\lambda_k = 1.$$

Then

$$\begin{aligned}
f(z) &= \sum_{k=0}^{\infty}\lambda_k f_k(z) = \lambda_0 f_0(z) + \sum_{k=1}^{\infty}\lambda_k f_k(z)\\
&= \left(1-\sum_{k=1}^{\infty}\lambda_k\right)\frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}\lambda_k\left\{\frac{a_{-p}}{z^p} - \frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+k+1)^{[}\beta(k+1)-(1-\alpha)]}z^k\right\}\\
&= \frac{a_{-p}}{z^p} - \sum_{jk=1}^{\infty}\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}\ z^k
\end{aligned} \tag{4.11}$$

Since

$$\sum_{k=1}^{\infty}(p+k+1)^n[\beta(k+1)-(1-\alpha)]\cdot\frac{(\beta(p-1)+1-\alpha)a_{-p}\lambda_k}{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}$$
$$=(\beta(p-1)+1-\alpha)a_{-p}\sum_{k=1}^{\infty}\lambda_k=(\beta(p-1)+1-\alpha)a_{-p}(1-\lambda_0)$$
$$\leq(\beta(p-1)+1-\alpha)a_{-p}, \tag{4.12}$$

by Theorem 2, $f(z)\in\sigma_p(n,\alpha,\beta)$.

Conversely, we suppose that $f(z)$ defined by (1.7) is in the class $\sigma_p(n,\alpha,\beta)$. Then by using (2.6), we get

$$a_k\leq\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}\quad(k\geq 1). \tag{4.13}$$

Setting

$$\lambda_k=\frac{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}{(\beta(p-1)+1-\alpha)a_{-p}}\ a_k\ (k\geq 1) \tag{4.14}$$

and

$$\lambda_0=1-\sum_{k=1}^{\infty}\lambda_k \tag{4.15}$$

we have (4.10). this completes the proof of Theorem 6.

5. Integral Operators

In this section we consider integral transforms of fucntions in the class $\sigma_p(n,\alpha,\beta)$.

Theorem 7 : Let the function $f(z)$ defined by (1.7) be in the class $\sigma_p(n,\alpha,\beta)$, then the integral transforms.

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)du\quad(0<u\leq 1,0<c<\infty) \tag{5.1}$$

are in $\sigma_p(n,\delta,\beta)$, where

$$\delta(\alpha,\beta,c,p)=\frac{(c+p+1)[2\beta-(1-\alpha)]-(2\beta-1)(\beta(p-1)+1-\alpha)c}{(c+p+1)[2\beta-(1-\alpha)]+\beta(p-1)+1-\alpha)c}. \tag{5.2}$$

The result is sharp for the function

$$f_j(z)=\frac{a_{-p}}{z^p}-\frac{(\beta(p-1)+1-\alpha)a_{-p}}{(p+2)^n(2\beta-1+\alpha)}\ z^j. \tag{5.3}$$

Proof : Let

$$F_{c+p-1}(z)=c\int_0^1 u^{c+p-1}f(uz)du=\frac{a_{-p}}{z^p}-\sum_{k=1}^{\infty}\frac{c}{c+p+k}a_kz^k. \tag{5.4}$$

In view of Theorem 2, it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{(p+k+1)^n[\beta(k+1)-(1-\delta)}{(\beta(p-1)+1-\alpha)a_{-p}} \ a_k \leq 1. \tag{5.5}$$

Since $f(z) \in \sigma_p(n, \delta, \beta)$, we have

$$\sum_{k=1}^{\infty} \frac{(p+k+1)^n[\beta(k+1)-(1-\alpha)]}{(\beta(p-1)+1-\alpha)a_{-p}} \ a_k \leq 1.$$

Thus (5.4) will be satisfied if

$$\frac{[\beta(k+1)-(1-\delta)]c}{(\beta(p-1)+1-\delta)(c+p+k)} \leq \frac{[\beta(k+1)-(1-\alpha)]}{(\beta(p-1)+1-\alpha)} \quad \text{for each} \quad k.$$

Solving for δ, we obtain

$$\delta \leq \frac{(c+p+k)[\beta(k+1)-(1-\alpha)]-[\beta(k+1)-1](\beta(p-1)+1-\alpha)c}{(c+p+k)[\beta(k+1)-(1-\alpha)]+(\beta(p-1)+1-\alpha)c} = F(k) \tag{5.7}$$

for each α, β, p and c fixed.

Then $F(k+1) - F(k) > 0$ for each k.

Hence $F(k)$ is an increasing function of k. Since

$$F(1) = \frac{(c+p+1)[2\beta-(1-\alpha)]-(2\beta-1)(\beta(p-1)+1-\alpha)c}{(c+p+1)[2\beta-(1-\alpha)]+(\beta(p-1)+1-\alpha)c}$$

the result follows.

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