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Alternating Group Iterative Method for Convection-Diffusion Equations

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Abstract

In this paper, a symmetry implicit scheme for solving convection-diffusion equations is presented with a kind of exponential type transformation. Based on the scheme a class of alternating group explicit iterative method(AGI) is constructed. The AGI method is convergent, unconditionally stable, and suitable for parallel computation. In order to verify the conclusions for the AGI method, several numerical examples are presented.

Mathematics Subject Classification: 65M06, 74S20

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1 Preface

We consider the following time-dependent initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + k\dfrac{\partial u}{\partial x} = \varepsilon \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad 0 \le t \le T \\ u(x,0) = f(x), \\ u(0,t) = g_1(t), u(1,t) = g_2(t). \end{cases} \tag{1.1}$$

Many numerical methods for solving convection-diffusion equations have been presented so far, which are sorted by explicit and implicit methods in general. Considering the stability and accuracy of explicit schemes and the computation difficulty of implicit schemes, it is a important work to construct numerical methods with absolute stability while easy to compute. With the development of parallel computer, researches on parallel numerical methods are getting more and more popular. D. J. Evans and A. R. B. Abdullah presented a class of alternating group method (AGE) for diffusion equations by the specifical combination of several asymmetric schemes in [1], and applied the method to convection-diffusion equations in [2]. The AGE method is widely cared for it is simple for computing, unconditionally stable, and has the property of parallelism. Based on the AGE method, many alternating group methods have been presented such as in[3-7]. Most of the methods inherit the advantages of the AGE method, and are of higher accuracy than the AGE method, But researches on alternating group iterative methods are scarcely presented.

We organize this paper as follows: In section 2, we present a symmetry implicit scheme for solving (1.1) at first, based on which a class of alternating group explicit iterative method(AGI) is constructed. In section 3, convergence analysis and stability analysis are given. In section 4, results of numerical experiments are presented.

2 The Alternating Group Iterative Method(AGI)

The domain $\Omega : (0,1) \times (0,T)$ will be divided into $(m \times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, t_n), $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_{m-1}^n)^T$.

The equation (1.1) is equivalent to $e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial t} = \varepsilon\frac{\partial}{\partial x}(e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial x})$. Integral from $x_{i-\frac{1}{2}}$ to $x_{i+\frac{1}{2}}$ we have $(\frac{\partial u}{\partial t})_i^{n+\frac{1}{2}}\int_{x_{i-\frac{1}{2}h}}^{x_{i+\frac{1}{2}h}} e^{-\frac{kx}{\varepsilon}}dx \approx \varepsilon[e^{-\frac{kh}{2\varepsilon}}(\frac{\partial u}{\partial x})_{i+\frac{1}{2}}^{n+\frac{1}{2}} - e^{\frac{kh}{2\varepsilon}}(\frac{\partial u}{\partial x})_{i-\frac{1}{2}}^{n+\frac{1}{2}}]$.

We can derive an implicit scheme for solving (1.1) as below:

$$(e^{\frac{kh}{2\varepsilon}}-e^{-\frac{kh}{2\varepsilon}})\frac{u_i^{n+1}-u_i^n}{\tau} = \frac{k}{h}[e^{-\frac{kh}{2\varepsilon}}(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{2}+\frac{u_{i+1}^n-u_i^n}{2})-e^{\frac{kh}{2\varepsilon}}(\frac{u_i^{n+1}-u_{i-1}^{n+1}}{2}+\frac{u_i^n-u_{i-1}^n}{2})]$$

Applying Taylor's formula to the scheme at $(x_i, t_{n+\frac{1}{2}})$, we can easily have that the truncation error of the scheme is $O(\tau^2 + h^2)$.

Let $p = e^{-\frac{kh}{2\varepsilon}}, q = e^{\frac{kh}{2\varepsilon}}, r = \frac{k\tau}{h(q-p)}$, then we have :

$$-\frac{rq}{2}u_{i-1}^{n+1}+[1+\frac{r}{2}(p+q)]u_i^{n+1}-\frac{rp}{2}u_{i+1}^{n+1} = \frac{rq}{2}u_{i-1}^n+[1-\frac{r}{2}(p+q)u_i^n+\frac{rp}{2}u_{i+1}^n \quad (2.1)$$

We denote (2.1) as $AU^{n+1} = F^n$. here

$$F^n = (2I - A)U^n + [\frac{rq}{2}(u_0^{n+1} + u_0^n), 0, \cdots, 0, \frac{rp}{2}(u_m^{n+1} + u_m^n)]^T$$

$$A = \begin{pmatrix} 1+\frac{r}{2}(p+q) & -\frac{rp}{2} & & & \\ -\frac{rq}{2} & 1+\frac{r}{2}(p+q) & -\frac{rp}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{rq}{2} & 1+\frac{r}{2}(p+q) & -\frac{rp}{2} \\ & & & -\frac{rq}{2} & 1+\frac{r}{2}(p+q) \end{pmatrix}_{(m-1)\times(m-1)}$$

The alternating group iterative method will be constructed in two conditions as follows: (1)$m = 4s+1,\ s$ is an integer. Let $A = \frac{1}{2}(G_1 + G_2)$, here

$$G_1 = \begin{pmatrix} G_{11} & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & G_{11} \end{pmatrix}_{(m-1)\times(m-1)}, \quad G_2 = \begin{pmatrix} G_{21} & & & & \\ & G_{11} & & & \\ & & \cdots & & \\ & & & G_{11} & \\ & & & & G_{21} \end{pmatrix}_{(m-1)\times(m-1)}$$

$$G_{11} = \begin{pmatrix} 1+\frac{r}{2}(p+q) & -\frac{rp}{2} & 0 & 0 \\ -\frac{rq}{2} & 1+\frac{r}{2}(p+q) & -rp & 0 \\ 0 & -rq & 1+\frac{r}{2}(p+q) & -\frac{rp}{2} \\ 0 & 0 & -\frac{rq}{2} & 1+\frac{r}{2}(p+q) \end{pmatrix}$$

$$G_{21} = \begin{pmatrix} 1+\frac{r}{2}(p+q) & -\frac{rp}{2} \\ -\frac{rq}{2} & 1+\frac{r}{2}(p+q) \end{pmatrix}$$

Then the alternating group iterative method I can be derived as below:

$$\begin{cases} (\rho I + G_1)U^{n+1(k+\frac{1}{2})} = (\rho I - G_2)U^{n+1(k)} + 2F^n \\ (\rho I + G_2)U^{n+1(k+1)} = (\rho I - G_1)U^{n+1(k+\frac{1}{2})} + 2F^n \end{cases} \tag{2.2}$$

Here k is the iterative number.

(2)$m = 4s+3,\ s$ is an integer. Let $A = \frac{1}{2}(\overline{G}_1 + \overline{G}_2)$, here

$$\overline{G}_1 = \begin{pmatrix} G_{11} & & & & \\ & \cdots & & & \\ & & \cdots & & \\ & & & G_{11} & \\ & & & & G_{21} \end{pmatrix}_{(m-1)\times(m-1)}, \quad \overline{G}_2 = \begin{pmatrix} G_{21} & & & & \\ & G_{11} & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & G_{11} \end{pmatrix}_{(m-1)\times(m-1)}$$

Then the alternating group iterative method II can be derived as below:

$$\begin{cases} (\rho I + \overline{G}_1)U^{n+1(k+\frac{1}{2})} = (\rho I - \overline{G}_2)U^{n+1(k)} + 2F^n \\ (\rho I + \overline{G}_2)U^{n+1(k+1)} = (\rho I - \overline{G}_1)U^{n+1(k+\frac{1}{2})} + 2F^n \end{cases} \tag{2.3}$$

3 Convergence Analysis and Stability Analysis

In order to verify the convergence of the AGI method, from [8] we have the following lemmas:
Lemma 1 Let $\theta>0$, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\|(\theta I+G)^{-1}\|_2 \le \theta^{-1} \tag{3.1}$$

Lemma 2 On the conditions of Lemma 1, we have:

$$\|(\theta I-G)(\theta I+G)^{-1}\|_2 \le 1 \tag{3.2}$$

Theorem 1 The alternating group iterative method I given by (2.2) is convergent.

proof: Let $\widehat{G}_1=\frac{1}{r}(G_1-I)$, $\widehat{G}_2=\frac{1}{r}(G_2-I)$, then $G_1=I+r\widehat{G}_1$, $G_2=I+r\widehat{G}_2$

From the construction of the matrixes $\widehat{G}_1$ and $\widehat{G}_2$ we can see they are both nonnegative definite real matrixes. Thus G_1, G_2, $(G_1+G_1^T)$, $(G_2+G_2^T)$ are all nonnegative matrixes. Then we have $\|(\rho I-G_1)(\rho I+G_1)^{-1}\|_2\le 1$, $\|(\rho I-G_2)(\rho I+G_2)^{-1}\|_2\le 1$.

From (2.2), we can obtain $U^{n+1}=GU^n+2(\rho I+G_2)^{-1}[(\rho I-G_1)(\rho I+G_1)^{-1}F^n+F^n]$, $G=(\rho I+G_2)^{-1}(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)$ is the growth matrix.

Let $\widetilde{G}=(\rho I+G_2)G(\rho I+G_2)^{-1}=(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)(\rho I+G_2)^{-1}$, then $\rho(G)=\rho(\widetilde{G})\le\|\widetilde{G}\|_2\le 1.$, which shows the alternating group method I given by (2.2) is convergent.

Analogously we have:

Theorem 2 The alternating group iterative method II given by (2.3) is also convergent.

We will use the Fourier method to analyze the stability of (2.1). Let $U^n=V^n e^{i\alpha x_j}$, then from (2.1) we have

$$V^{n+1}=V^n\frac{1-\frac{r}{2}(p+q)+\frac{r}{2}(p+q)cos(\alpha h)+i\frac{r}{2}(p-q)sin(\alpha h)}{1+\frac{r}{2}(p+q)-\frac{r}{2}(p+q)cos(\alpha h)-i\frac{r}{2}(p-q)sin(\alpha h)}. \tag{3.3}$$

Let $x=\frac{r}{2}(p+q)-\frac{r}{2}(p+q)cos(\alpha h)$, $y=\frac{r}{2}(p-q)sin(\alpha h)$, then we have

$$V^{n+1}=\frac{1-x+iy}{1+x-iy}V^n$$

Considering $x\ge 0$, it follows that $|\frac{1-x+iy}{1+x-iy}|^2=\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\le 1$, so we have:

Theorem 3 The scheme (2.1) is unconditionally stable.

4 Numerical Experiments

We consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + k\dfrac{\partial u}{\partial x} = \varepsilon \dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 \le t \le T \\ u(x,0) = 0, \\ u(0,t) = 0, u(1,t) = 1. \end{cases} \tag{4.1}$$

The exact solution of the problem above is denoted in [2] as below:

$$u(x,t) = \frac{e^{\frac{kx}{\varepsilon}} - 1}{e^{k\varepsilon} - 1} + \sum_{n=1}^{\infty} \frac{(-1)^n n\pi}{(n\pi)^2 + (\frac{k}{2\varepsilon})^2} e^{\frac{k(x-1)}{2\varepsilon}} sin(n\pi x) e^{-[(n\pi)^2\varepsilon + \frac{k^2}{4\varepsilon}]t}$$

Let A.E denote absolute error, while P.E denote relevant error. A.E=$|u_i^n - u(x_i,t_n)|$, P.E=$100 \times |u_i^n - u(x_i,t_n)/u(x_i,t_n)|$. In the numerical experiments we let $k = \varepsilon = \rho = 1$. Using the iterative error 1×10^{-6} to control the process of iterativeness, We will present the numerical results in the following tables:

Table 1: The numerical results for the iterative scheme I

	$m = 17, \tau = 10^{-4}, t = 100\tau$	$m = 17, \tau = 10^{-5}, t = 100\tau$
A.E	6.513×10^{-5}	7.133×10^{-5}
P.E	3.106×10^{-2}	8.477×10^{-2}
average iterative times	3	2

Table 2: The numerical results for the iterative scheme I

	$m = 21, \tau = 10^{-4}, t = 100\tau$	$m = 21, \tau = 10^{-4}, t = 1000\tau$
A.E	4.269×10^{-5}	1.718×10^{-5}
P.E	2.084×10^{-2}	1.127×10^{-2}
average iterative times	3	2.994

Table 3: The numerical results for the iterative scheme II

	$m = 19, \tau = 10^{-4}, t = 100\tau$	$m = 19, \tau = 10^{-5}, t = 100\tau$
A.E	5.211×10^{-5}	5.707×10^{-6}
P.E	2.521×10^{-2}	1.561×10^{-2}
average iterative times	3	2

Table 4: The numerical results for the iterative scheme II

	$m = 23, \tau = 10^{-4}, t = 100\tau$	$m = 23, \tau = 10^{-4}, t = 1000\tau$
A.E	3.563×10^{-5}	1.434×10^{-5}
P.E	1.751×10^{-2}	9.405×10^{-3}
average iterative times	3	2.998

From Table 1-4 we can see that the numerical solution for the AGI method (2.2) and (2.3) can converge to the exact solution fast, and we can get higher accuracy when the spatial step diminishes, which accords to the conclusion of convergence and error analysis.

5 Conclusions

In this paper, an universal alternating group iterative(AGI) method is derived by using a special implicit scheme of high accuracy, also convergence analysis and stability analysis are finished. The AGI method is convenient to use in solving large equation set, and is suitable for parallel computation. Furthermore, using the Taylor formula we can obtain another implicit scheme of higher accuracy than (2.1), and based on the scheme we can derive another alternating group iterative method by almost the same procession in section 2. The construction of the AGI method can also be applied to other partial differential equations.

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