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A New Method of Solving Singular Multi-pantograph Delay Differential Equation in Reproducing Kernel Space

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Abstract

In this paper, we present a new method for solving singular two-point boundary value problem for multi-pantograph delay differential equation having singular coefficients. Its exact solution is represented in the form of series in reproducing kernel space. In the mean time, the n-term approximation $u_n(x)$ to the exact solution $u(x)$ is obtained. An numerical example is studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

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1 Introduction

In this paper, we consider the following inhomogeneous second order multi-pantograph delay differential equation with singularities in reproducing kernel space

$$\begin{cases} u''(x) + \sum\limits_{i=1}^{m} \frac{1}{p_i(x)} u'(h_i x) + \frac{1}{q(x)} u(x) = g(x), \quad 0 < x, h_i < 1, i = 1, \dots, m \\ u(0) = 0, u(1) = 0, \end{cases} \tag{1.1}$$

where $u(x) \in W_2^3[0,1]$, $g(x) \in W_2^1[0,1]$, $p_i(x)$, $q(x)$ are continuous and maybe equal to zero at 0 or 1. These equations arise in a variety of applications , such as numeber theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structure, quantum mechanics and cell growth. Therefore, the problem has attracted much attention and has been studied by many authors [1],[3-9],[10]. In recent years, much work has been done in reproducing kernel space[2],[8],[11]. The basic motivation of this work is to apply a new method to solve multi-pantograph delay differential equation in reproducing kernel space.

In this paper, we will give the representation of exact solution to $Eq.(1.1)$ and approximate solution in the reproducing kernel space under the assumption that the solution to $Eq.(1.1)$ is unique. The approach is simple and effective. If the solution to $Eq.(1.1)$ is not unique, then the solution we obtain is a least-norm solution.

After multiplying (1.1) by $\prod_{i=1}^{m} p_i(x)q(x)$, we find that

$$\begin{cases} \prod_{i=1}^{m} p_i(x)q(x)u''(x) + q(x)\sum_{i=1}^{m} u'(h_i x) + \prod_{i=1}^{m} p_i(x)u(x) = \prod_{i=1}^{m} p_i(x)q(x)g(x), \\ 0 < x, h_i < 1, i = 1, \ldots, m \\ u(0) = 0, u(1) = 0, \end{cases} \tag{1.2}$$

Define operator L as follows:

$Lu = \prod_{i=1}^{m} p_i(x)q(x)u''(x) + q(x)\sum_{i=1}^{m} u'(h_i x) + \prod_{i=1}^{m} p_i(x)u(x)$

and let $f(x) = \prod_{i=1}^{m} p_i(x)q(x)g(x)$. Then $Eq.(1.1)$ can be convert into following form

$$\begin{cases} Lu = f(x), x \in [0,1] \\ u(0) = 0, u(1) = 0, \end{cases} \tag{1.3}$$

where $f \in W_2^1[0,1], u \in W_2^3[0,1]$. $W_2^1[0,1]$ and $W_2^3[0,1]$ are defined in the following section.

2 Several Reproducing Kernel Spaces

1 The reproducing kernel space $W_2^3[0,1]$

The inner product space $W_2^3[0,1]$ is defined as $W_2^3[0,1] = \{u(x) \mid u, u', u''$ are absolutely continuous real value functions, $u, u', u'', u^{(3)} \in L^2[0,1], u(0) = 0, u(1) = 0\}$. The inner product in $W_2^3[0,1]$ is given by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u^{(3)}v^{(3)}dy, \tag{2.1}$$

and the norm $\| u \|_{W_2^3}$ is denoted by $\| u \|_{W_2^3}= \sqrt{(u,u)_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

Theorem 2.1. *The space $W_2^3[0,1]$ is a reproducing kernel space. That is, for any $u(y) \in W_2^3[0,1]$ and each fixed $x \in [0,1]$, there exists $R_x(y) \in W_2^3[0,1]$, $y \in [0,1]$, such that $(u(y), R_x(y))_{W_2^3} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} \frac{-\left((-1+x)\,y\left(6\,x^2\,y-4\,x^3\,y+x^4\,y+y^4+x\left(120-120\,y-5\,y^3+y^4\right)\right)\right)}{120}, & y \le x, \\ \frac{-\left(x\,(-1+y)\left(120\,y-5\,x^3\,y+x^4\,(1+y)+x\,y\left(-120+6\,y-4\,y^2+y^3\right)\right)\right)}{120}, & y > x. \end{cases} \tag{2.2}$$

The proof of Theorem (2.1) see for [2,8].

2 The reproducing kernel space $W_2^1[0,1]$

The inner product space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u(x) \mid u$ is absolutely continuous real value function, $u, u' \in L^2[0,1]\}$. The inner product and norm in $W_2^1[0,1]$ are given respectively by

$$(u(x), v(x))_{W_2^1} = \int_0^1 (uv + u'v')dx, \ \| u \|_{W_2^1}= \sqrt{(u,u)_{W_2^1}},$$

where $u(x), v(x) \in W_2^1[0,1]$. In Ref.[11], the authors had proved that $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel is

$$\overline{R}_x(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1) + \cosh(|x-y|-1)].$$

3 The solution of Eq.(1.3)

In this section, the solution of $Eq.(1.3)$ is given in the reproducing kernel space $W_2^3[0,1]$.

In $Eq.(1.3)$, it is clear that $L : W_2^3[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where L^* is the adjoint operator of L . The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^\infty$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^\infty$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1,2,...). \tag{3.1}$$

Theorem 3.1. *For $Eq.(1.3)$, if $\{x_i\}_{i=1}^\infty$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^\infty$ is the complete system of $W_2^3[0,1]$ and $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.*

Proof. We have

$$\begin{aligned} \psi_i(x) &= (L^*\varphi_i)(x) = ((L^*\varphi_i)(y), R_x(y)) \\ &= (\varphi_i(y), L_y R_x(y)) = L_y R_x(y)|_{y=x_i}. \end{aligned}$$

The subscript y by the operator L indicates that the operator L applies to the function of y.
Clearly, $\psi_i(x) \in W_2^3[0,1]$.
For each fixed $u(x) \in W_2^3[0,1]$, let $(u(x), \psi_i(x)) = 0, (i = 1, 2, ...)$, which means that,

$$(u(x), (L^*\varphi_i)(x)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(x_i) = 0. \tag{3.2}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of the Theorem 3.1 is complete. □

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of Eq.(1.3) is unique, then the solution of Eq.(1.3) is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{3.3}$$

Proof. Applying Theorem 3.1, it is easy to know that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^3[0,1]$.
Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0,1]$, hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}(u(x), \overline{\psi}_i(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(u(x), L^*\varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(Lu(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(f(x), \varphi_k(x))\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \end{aligned} \tag{3.4}$$

So, the proof of the theorem is complete. □

Now, the approximate solution $u_n(x)$ can be obtained by the n-term intercept of the exact solution $u(x)$ and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi}_i(x). \tag{3.5}$$

Theorem 3.3. *Assume $u(x)$ is the solution of Eq.(1.3) and $r_n(x)$ is the error between the approximate $u_n(x)$ and the exact solution $u(x)$. Then the error $r_n(x)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^3}$.*

Proof. From (3.4), (3.5), it follows that

$$\begin{aligned} \| r_n \|_{W_2^3} &= \| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k) \overline{\psi}_i(x) \|_{W_2^3} \\ &= \sum_{i=n+1}^{\infty} (\sum_{k=1}^{i} \beta_{ik} f(x_k))^2. \end{aligned} \tag{3.6}$$

(3.6) shows that the error r_n is monotone decreasing in the sense of $\| \cdot \|_{W_2^3}$. The proof is complete. □

4 Numerical example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 5.0. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement with each other.

Example 1

Consider equation

$$\begin{cases} u''(x) + \frac{1}{x}u'(\frac{1}{2}x) + \frac{1}{x^2}u'(\frac{1}{4}x) + \frac{1}{1-x}u(x) = f(x), 0 < x \leq 1, \\ u(0) = 1, u(1) = e, \end{cases}$$

where $f(x) = \frac{-\left(e^{\frac{x}{4}}\,(-1+x)\right)}{4} - \frac{e^{\frac{x}{2}}\,(-1+x)\,x}{2} - e^x\,(-2+x)\,x^2$. The true solution is e^x. Using our method, we choose 20 points and 100 points on $[0, 1]$ respectively. The numerical results are given in the following table 1, 2.

Table 1: Numerical results for example 1($n = 26$).

x	True solution u(x)	Approximate solution u_{20}	Absolute error
0.001	1.001	1.001	3.8E-9
0.08	1.08329	1.08328	4.8E-06
0.16	1.17351	1.17349	2.1E-05
0.32	1.37713	1.37709	3.4E-05
0.48	1.61607	1.61605	2.5E-05
0.64	1.89648	1.89647	7.3E-06
0.80	2.22554	2.22555	1.1E-05
0.96	2.6117	2.61172	2.6E-05
1.00	2.71828	2.71828	0

Table 2: Numerical results for example 1($n = 51$).

x	True solution u(x)	Approximate solution u_{100}	Absolute error
0.001	1.001	1.001	3.1E-10
0.08	1.08329	1.08329	2.6E-07
0.16	1.17351	1.17351	1.8E-06
0.32	1.37713	1.37713	4.1E-06
0.48	1.61607	1.61608	4.7E-06
0.64	1.89648	1.89649	4.3E-06
0.80	2.22554	2.22554	3.0E-06
0.96	2.6117	2.6117	1.0E-06
1.00	2.71828	2.71828	0

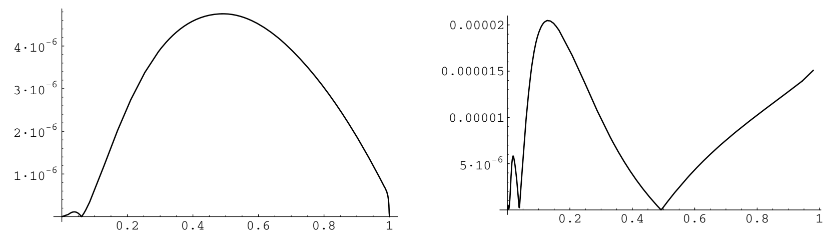

Figure 1: Figure of absolute errors $|u - u_{100}|$ and $|u' - u'_{100}|$ for Example 5.1

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