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The Dimensions of the Level Sets of the Generalized Iterated Brownian Motion

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Abstract

Let $\{W_1(t) : t \geq 0\}$ and $\{W_2(t) : t \geq 0\}$ be two independent Brownian motions with $W_1(0) = W_2(0) = 0$. $H = \{W_1(|W_2(t)|), t \geq 0\}$ is called a generalized iterated Brownian motion. In this paper, the Hausdorff dimension and packing dimension of the level sets

$$\{t \in [0, 1], H(t) = x\}$$

are established.

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1. Introduction and results

Let $\{W_1(t),\ 0 \leq t < \infty\}$ and $\{W_2(t),\ t \geq 0\}$ be two independent Wiener processes. Define the iterated Brownian motion

$$Y \triangleq \{W_1(W_2(t)), \qquad t \geq 0\}$$

and a modification of the iterated Brownian motion

$$H \triangleq \{W_1(|W_2(t)|), \quad t \geq 0\}.$$

The latter was used by Funaki (1979) to give a probabilistic solution to the partial differential equation

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{8} \cdot \frac{\partial u}{\partial t} \quad \text{with} \;\; u(0, x) = u_0(x).$$

$\{H(t),\ 0 \le t < \infty\}$ is called a generalized iterated Brownian motiom (GIBM). Csáki et al (1996) gave a detailed study of the local time process $\{L^*(x,t); x \in R^1, t \ge 0\}$ of $H(t)$ and established a result on continuity modulus in t.

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{x\in R} \frac{|L^*(x,t+h) - L^*(x,t)|}{2^{7/4} h^{3/4} (\log h^{-1})^{5/4}} = 1 \qquad \text{a.s.}$$

Eisenbaum and Földes (2001) have given strong approximations of continuous additive functional of a Brownian motion by a GIBM.

Recently, there has been a lot of investigations on sample path properties of an iterated Brownian motion. But for a GIBM, we don't find more investigations.

In this paper, we use arguments based on local time to obtain the Hausdorff dimension and the packing dimension of the level set $E(x,T) = \{t \in [0,T], H(t) = x\}$. Our main results are as follows.

Theorem 1.1 With probability one,

$$\dim_H E(x,T) = \frac{3}{4} \tag{1.1}$$

for all $0 < T \le 1$ and every x in the interior of the range of $\{H(t), t \in [0,T]\}$.

Theorem 1.2 With probability one,

$$\dim_P E(x,T) = \frac{3}{4} \tag{1.2}$$

for all $0 < T \le 1$ and every x in the interior of the range of $\{H(t), t \in [0,T]\}$.

Throughout this paper, the Hausdorff dimension and the packing dimension of A are denoted by $\dim_H A$ and $\dim_p A$, respectively.

2. The Hausdorff dimension of the level set

The following lemmas are the key tools to prove our theorems.

Lemma 2.1 (Csáki et al (1996)) Let $\{L^*(x,t); x \in R, t \ge 0\}$ be the local time of $H(t)$, then it is jointly continuous in x and t a.s. And

$$\limsup_{h\to 0} \sup_{0\le t\le 1} \sup_{x\in R} \frac{|L^*(x,t+h) - L^*(x,t)|}{2^{7/4} h^{3/4} (\log h^{-1})^{5/4}} \le 1 \qquad \text{a.s.} \tag{2.1}$$

The following is the lower bound of the Hausdorff dimension of the level set.

Theorem 2.1 With probability one,

$$\dim_H E(x,T) \ge \frac{3}{4} \quad \text{a.s.} \tag{2.2}$$

Proof Let F_x be the measure on $[0,T]$ induced by the local time $L^*(x,\cdot)$, i.e., define F_x on $[0,T]$ by setting

$$F_x(t,t+h) = L^*(x,t+h) - L^*(x,t).$$

It is easy to see that F_x is completely additive measure defined on real Borel sets. By (2.1)

$$\frac{F_x(t,t+h)}{h^{3/4-\epsilon}} \leq K < \infty, \tag{2.3}$$

for any $\epsilon > 0$. Applying the principle of the mass distribution, we have

$$\phi_1 - m(E) \geq K^{-1} F_x(E) \geq 0 \tag{2.4}$$

where

$$\phi_1(h) = h^{3/4-\epsilon}.$$

Hence we have

$$\dim_H E(x,T) \geq \frac{3}{4} - \epsilon \quad \text{a.s.}$$

By arbitrariness of ϵ, we obtain (2.2).

In order to prove the upper bound we will need two lemmas.

Lemma 2.4

$$\limsup_{h\to 0} \frac{H(t+h)-H(t)}{h^{1/4}(\log\log\frac{1}{h})^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{2.5}$$

Proof Csáki et al (1989) proved the following result: If $L_2(0,t)$ is the local time at zero of $W_2(t)$, and $U(t) \triangleq W_1(L_2(0,t))$, then

$$\limsup_{t\to\infty} \frac{U(t)}{t^{1/4}(\log\log t)^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{2.6}$$

Moreover we have

$$\{L_2(0,t); t \geq 0\} \stackrel{d}{=} \{\sup_{s\leq t} W_2(s); t\geq 0\} \stackrel{d}{=} \{|W_2(t)|; t \geq 0\}, \tag{2.7}$$

which is the well-known Lévy's theorem, and can be find in Kallenberg (2001). By (2.7) and scaling property of the Brownian motion,

$$U(t) \stackrel{d}{=} W_1(|W_2(t)|) \stackrel{d}{=} W_1(|tW_2(t^{-1})|) \stackrel{d}{=} t^{1/2} W_1(|W_2(t^{-1})|),$$

which, by letting $h = 1/t$, implies $U(h) = h^{-1/2}U(t)$ and hence

$$\limsup_{h\to 0} \frac{U(h)}{h^{1/4}(\log\log h^{-1})^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.} \tag{2.8}$$

from (2.6). By (2.7) again,

$$\begin{aligned} H(t+h)-H(t) &= W_1(|W_2(t+h)|)-W_1(|W_2(t)|) \\ &\stackrel{d}{=} W_1(|W_2(t+h)|-|W_2(t)|) \\ &\stackrel{d}{=} W_1(L_2(0,t+h)-L_2(0,t)) \\ &\stackrel{d}{=} W_1(L_2(0,h)) \\ &= U(h). \end{aligned}$$

Then (2.5) follows from (2.8).

Lemma 2.5 Let $F:[0,1]\to R$ satisfy the uniform Hölder condition of order $\gamma,\ 0<\gamma<1$, and possess a jointly continuous local time L, then for every real x

$$\dim_H[t\in[0,1];F(t)=x]\le 1-\gamma.$$

Proof This is an analog to Lemma 7 of Adler (1978).

Theorm 2.2 With probability one,

$$\dim_H E(x,T)\le\frac{3}{4}\quad \text{a.s.}$$

Proof By Lemma 2.4, for any $\epsilon>0$, $0<\epsilon<\frac{1}{4}$, and h small enough, we have

$$H(t+h)-H(t)\le ch^{\frac{1}{4}-\epsilon}\qquad \text{a.s.}$$

which, in combination with Lemma 2.5 and arbitrariness of ϵ, implies the conclusion of the theorem.

Now, putting together Theorem 2.1 and Theorem 2.2, we conclude Theorem 1.1.

3. The packing dimension of the level set

In this section, we will prove Theorem 1.2. The packing dimension of any set always does not smaller than its Hausdorff dimension, so we just prove the upper bound of the packing dimension of the level set is $\frac{3}{4}$. Here we need some lemmas.

The next result is the well-known Chung's law of LIL for the Brownian motion. An analogy for the local time of the Brownian motion is immediate.

Lemma 3.1 Let $\{W(t),0\le t\le 1\}$ be a Brownian motion, and $L(x,t)$ be its local time, then

$$\liminf_{t\to 0}\sup_{0\le r\le t}\frac{|W(r)|}{(t/\log\log t^{-1})^{1/2}}=\frac{\pi}{2\sqrt{2}}\qquad \text{a.s.} \tag{3.1}$$

and

$$\liminf_{t\to 0} \sup_{x\in R} \frac{L(x,t)}{(t/\log\log t^{-1})^{1/2}} = \frac{\pi}{2\sqrt{2}} \qquad \text{a.s.} \tag{3.2}$$

Lemma 3.2 Let $L^*(x,t)$ be the local time of GIBM $H(t)$, then we have

$$\liminf_{h\to 0} \sup_{0\le t\le 1} \sup_{x\in R} \frac{L^*(x,t+h)-L^*(x,t)}{h^{3/4}(\log h^{-1})^{1/2}} \ge C \qquad \text{a.s.}$$

for some $C > 0$.

Proof Let $L_2'(x,t)$ be the local time of $|W_2(t)|$ and $L_1(x,t)$ be the local time of $W_1(t)$. By Csáki et.al. (1996),

$$L^*(x,t) = \int_0^\infty L_2'(s,t) d_s L_1(x,s)$$

and by (3.2), for any $\epsilon > 0$

$$\sup_{0\le t\le 1-h} \sup_{x\in R} (L_2'(x,t+h) - L_2'(x,t)) \ge 2(1-\epsilon)(2h\log h^{-1})^{\frac{1}{2}} \qquad \text{a.s.}$$

provided h is small enough for $x \ge 0$. Where we used the fact that

$$L_2'(x,t) = L_2(x,t) + L_2(-x,t) = 2L_2(x,t) \qquad \text{a.s.} \tag{3.3}$$

So we have

$$\begin{aligned}
&\sup_{0\le t\le 1-h} \sup_{x\in R} (L^*(x,t+h) - L^*(x,t)) \\
&\quad = \sup_{0\le t\le 1-h} \sup_{x\in R} \int_0^\infty (L_2'(s,t+h) - L_2'(s,t)) d_s L_1(x,s) \\
&\quad \ge \sup_{0\le t\le 1-h} \sup_{x\in R} \int_0^{A(h)} (L_2'(s,t+h) - L_2'(s,t)) d_s L_1(x,s)
\end{aligned}$$

$$\ge 2(1-\epsilon)(2h\log h^{-1})^{\frac{1}{2}} \sup_{x\in R} (L_1(x, A(h)) \tag{3.4}$$

where $A(h) = \sup_{0\le r\le h} |W_2(r)|$.

Applying (3.1), we have

$$\liminf_{h\to 0} \sup_{0\le r\le h} \frac{A(h)}{(h/\log\log h^{-1})^{1/2}} = \frac{\pi}{2\sqrt{2}} \qquad \text{a.s.} \tag{3.5}$$

By $\lim_{h\to 0} A(h) = 0$ a.s. and (3.2)

$$\liminf_{h\to 0} \sup_{x\in R} \frac{L_1(x, A(h))}{[A(h)/\log\log A(h)^{-1})]^{1/2}} = \frac{\pi}{2\sqrt{2}} \qquad \text{a.s.} \tag{3.6}$$

Combining (3.4)-(3.6), we conclude Lemma 3.2.

Now we prove Theorem 1.2.

Proof By Theorem 19.1 of Kallenberg (2001), for each x, $L^*(x,\cdot)$ has its support contained in $\{t \in [0,1]; H(t) = x\}$, and $L^*(x,\cdot)$ is Borel measurable. By Lemma 3.2, for any $\epsilon,\ 0 < \epsilon < \frac{3}{4}$, we have

$$L^*(x,t+h) - L^*(x,t) \geq Ch^{3/4-\epsilon} \ ,$$

Applying the Lemma of Frostman (1935) , the packing measure

$$0 < \phi_1 - p(E(x,T)) < +\infty,$$

where $\phi_1(h) = h^{3/4-\varepsilon}$, then we have

$$\dim_p E(x,T) \leq \frac{3}{4} - \epsilon \quad a.s.$$

By arbitrariness of ϵ and Theorem 1.1, we complete the proof of Theorem 1.2.

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