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Melnikov Analysis of Chaos in an Epidemiological Model with Almost Periodic Incidence Rates

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Abstract

In this paper we study the well-known SIR model of epidemic dynamics with an almost periodic incidence rate. We make use of similar techniques utilized in a recent work to establish the existence of chaotic motions. To illustrate those chaotic motions, some numerical simulations are made.

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1 Introduction

In recent years, some simple mathematical models of epidemics have been used to gain some insight into the mechanisms, which are responsible of epidemic growths. At their most basic [1], these models assume a constant population, which is divided into three classes: S denote the susceptible people, I is the number of people infected , and R, the number of people who have recovered or who have the disease but are no longer infective. Such a model is known as the SIR model. Since the population is constant one can write:

$$S(t) + I(t) + R(t) = 1.$$

Now, under the assumption that those individuals who have had the disease

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are immune we obtain the well-known SIR model

$$\left\{\begin{array}{rcl} \dfrac{dS}{dt} & = & -B(I,t)S+\mu-\mu S, \\ \dfrac{dI}{dt} & = & B(I,t)S-(\gamma+\mu)I, \\ \dfrac{dR}{dt} & = & \gamma I-\mu R, \end{array}\right. \tag{1}$$

where the constants γ and μ are positive,while the incidence function $B(I,t)$ is usually taken as a periodic function in time, see e.g., [5] and [6]. More precisely, it is usual to take $B(I,t)=\beta(t)I$, where $\beta(t)$ (called the transmission rate) is either constant, or a periodic modulation about a constant value, for instance $\beta(t)=\beta_0(1+\beta_1 sin(\omega t))$. It is in particular well-known that when β_1 is sufficiently large, it is possible to obtain strong numerical evidence of a chaotic motion, see, e.g., [2], [9], and [12] and the references therein. Note also that some complicated versions of the incidence function had been studied in [8], [10], and [11]; there, the transmission rate is taken independent of time and $B(I,t)$ is proportional to a power of I. Chaotic behavior has been observed numerically in many numerical simulations of forced SIR models. Glendinning and Perry [6] have utilized a periodically forced nonlinear incidence rate to prove that there are SIR models with chaotic trajectories. In this paper we will consider an incidence function, which is (possibly not periodic) almost periodic in time, that is,

$$B(I,t)=\beta(t)I^p,\ p>1, \tag{2}$$

whose transmission rate is

$$\beta(t)=\frac{p^{2p}}{(p-1)^{2p-1}}\mu(1+\varepsilon^2 b_1+\varepsilon^4 b_2+\varepsilon^5 b_3\Gamma(\mu\varepsilon\Omega t)), \tag{3}$$

with $\Gamma:\mathbb{R}\mapsto\mathbb{R}$, a (possibly not periodic) almost periodic function, and b_1, b_2, b_3, Ω and p are constants such that

$$2pb_2-(p-1)b_1^2>0. \tag{4}$$

For the sake of simplicity, throughout the rest of the paper we suppose that $\Gamma(t)=\sin t+\sin\sqrt{2}t$. Note that Γ is an example of an almost periodic function, which is not periodic. Moreover, we set

$$\gamma=\frac{\mu(1+\varepsilon^2 b_1)}{p-1}. \tag{5}$$

Definition 1.1. *A continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that*

$$|f(t+\tau) - f(t)| < \varepsilon \ \textit{ for each } \ t \in \mathbb{R}.$$

The number τ above is called an ε-*translation* number of f, and the collection of all such functions will be denoted $AP(\mathbb{R})$. Among other things, every periodic function is almost periodic; the converse being false. If $f, g \in AP(\mathbb{R})$ and $\lambda \in \mathbb{R}$, then $f.g$ and $f + g$ belong to $AP(\mathbb{R})$. Furthermore, the functions defined by $t \mapsto f(\lambda t)$, $t \mapsto f(t+\lambda)$, and $t \mapsto \lambda f(t)$ are almost periodic. In addition to the above, each almost periodic function is (i) bounded, (ii) uniformly continuous, and (iii) its range is precompact in $\mathbb{R}$. For additional details on almost periodic functions we refer the reader to the landmark book by Corduneanu [4].

This paper is organized as follows: Section 2 describes the scaling of the SIR variables that will enable us to use the Melnikov's analysis [5, 6, 7, 13]. In Section 3 we prove that (1) has chaotic solutions for sufficiently large (but still order one compared to ε) values of b_3. To achieve this goal we apply Melnikov's method to the defining equations. In section 4 we give the numerical simulations of the chaotic conclusion.

2 Scaling the Equations

From the equation $S = 1 - I - R$ and since $\mu > 0$, defining a new timescale $t' = \mu t$, we have shown in [5] that the basic SIR model (1) can be written as follows:

$$\begin{cases} \dfrac{dI}{dt'} &= \beta'(t')I^p(1 - I - R) - (\gamma' + 1)I, \\ \\ \dfrac{dR}{dt'} &= \gamma' I - R, \end{cases} \tag{6}$$

where $\beta = \mu\beta'$ and $\gamma = \mu\gamma'$. Like in [5], our first aim is to make some changes of coordinates in order to adapt the system (6) to the Melnikov Analysis.
In the following we drop the primes in (6). So, defining a new variable $y = \gamma I - R$, (6) becomes

$$\begin{cases} \dot{R} &= y \\ \dot{y} &= -(1+\gamma)R - (2+\gamma)y \\ &+ \ \beta(t)\gamma^{-p+1}(R+y)^p(1 - \gamma^{-1}(R+y) - R). \end{cases} \tag{7}$$

Stationary points of (7) have $y = 0$, $R = 0$ which corresponds to the disease-free equilibrium; or $y = 0$ and R satisfies

$$-(1+\gamma)R + \beta\gamma^{-p+1}R^p(1 - \gamma^{-1}R - R) = 0. \tag{8}$$

Let $\sigma = \dfrac{1+\gamma}{\beta\gamma^{-p+1}}$, $H = \dfrac{1}{1+\gamma^{-1}}$, so (8) becomes

$$R^{p-1}\left(1 - \frac{1}{H}R\right) = \sigma, \tag{9}$$

where $0 < R < H$. Let

$$f(R) = R^{p-1}\left(1 - \frac{1}{H}R\right), \ R \in [0, H].$$

Then

$$f'(R) = R^{p-2}\left(p - 1 - \frac{P}{H}R\right). \tag{10}$$

If $R \in \left(0, \dfrac{p-1}{P}H\right)$ then $f'(R) > 0$; and if $R \in \left(\dfrac{p-1}{P}H, H\right)$ then $f'(R) < 0$.

Let

$$\sigma^* = f\left(\frac{p-1}{P}H\right) = \frac{(p-1)^{p-1}}{P^p}H^{p-1}.$$

Then σ^* is the maximum of $f(R)$ on $(0, H)$.

If $\sigma < \sigma^*$ then the system (7) has two positive equilibria R_1 and R_2; if $\sigma > \sigma^*$ then the positive equilibrium does not exist; if $\sigma = \sigma^*$ then there is one positive equilibrium.

Theorem 2.1. *If $\beta > \dfrac{p^p(1+\gamma)^p}{(p-1)^{p-1}}$, then the system (7) have two positive equilibria $(R_1, 0)$ and $(R_2, 0)$, which correspond to the infective equilibria; if $\beta < \dfrac{p^p(1+\gamma)^p}{(p-1)^{p-1}}$, then the positive equilibrium does not exist; if $\beta = \dfrac{p^p(1+\gamma)^p}{(p-1)^{p-1}}$, then there is one positive, which correspond to the infective equilibrium.*

So, the system (7) has a saddlenode bifurcation if

$$\beta = \beta_{sn} = \frac{p^p(1+\gamma)^p}{(p-1)^{p-1}}. \tag{11}$$

At this value, the equilibrium is

$$S_{sn} = \frac{1}{p}, \quad I_{sn} = \frac{p-1}{p(1+\gamma)}, \quad R_{sn} = \frac{(p-1)\gamma}{p(1+\gamma)}. \tag{12}$$

The Jacobian matrix of the system (7) at point $(R, 0)$ is

$$A = \begin{pmatrix} 0 & 1 \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$\begin{aligned} J_{21} &= (p-1)(1+\gamma) - \beta\gamma^{-p+1}(1+\gamma^{-1})R^p, \\ J_{22} &= p(1+\gamma) - (2+\gamma) - \beta\gamma^{-p}R^p. \end{aligned}$$

The determinant of this matrix $(-J_{21})$ vanishes identically at the saddlenode bifurcation, but if, in addition, the trace J_{22} also vanishes, then we have a more degenerate situation, known as Takens-Bogdanov point, and near this point it should be possible to write the equations in nearly Hamiltonian form [7]. The trace of the Jacobian matrix vanishes at the saddlenode if

$$\gamma = \frac{1}{p-1}; \quad \beta = \frac{p^{2p}}{(p-1)^{2p-1}}. \tag{13}$$

For these values of the parameters, the degenerate equilibrium is

$$S_{tb} = \frac{1}{p}, \quad I_{tb} = \left(\frac{p-1}{p}\right)^2, \quad R_{tb} = \frac{p-1}{p^2}.$$

Note that a curve of Hopf bifurcations also terminates at this point. This curve is given by $J_{22} = 0$; $J_{21} < 0$. To perturb about this degenerate point, we define new variables and parameters by

$$R = \frac{p-1}{p^2} + x; \quad \gamma = \frac{1}{p-1} + g; \quad \beta = \frac{p^{2p}}{(p-1)^{2p-1}}(1+b),$$

where x, g and b are to be thought of as small. Substituting into (7) and ignoring cubic orders and higher we find

$$\begin{cases} \dot{x} = y \\ \dot{y} = a_0 + a_1 x + a_2 y - \frac{1}{2}(p-1)^{-1}p^4x^2 - p^3xy \\ \qquad -\frac{p-2}{2(p-1)}p^3y^2 + O((|x|+|y|+|g|+|b|)^3), \end{cases} \tag{14}$$

where

$$\begin{aligned} a_0 &= -\frac{g^2}{2p}(p-2)(p-1)^3 - \frac{p-1}{p}g + \frac{b}{p} - \frac{(p-1)^2}{p^2}bg, \\ a_1 &= \frac{p}{p-1}(b + 2gp - 3gp^2 + gp^3), \\ a_2 &= \frac{1}{p-1}[b(2p-1) + g(p^4 - 4p^3 + 5p^2 - 3p + 1)]. \end{aligned}$$

We introduce a small parameter ε by setting

$$b = b_1\varepsilon^2 + b_2\varepsilon^4 + b_3\varepsilon^5 sin(\varepsilon\Omega t); \quad g = \frac{b_1\varepsilon^2}{p-1}. \tag{15}$$

In accordance with (3), so

$$\begin{aligned} a_0 &= \frac{2b_2p - b_1^2(p^3 - 3p^2 + 4p - 2)}{2p^2}\varepsilon^4 + \frac{b_3}{p}sin(\varepsilon\Omega t)\varepsilon^5 + O(\varepsilon^6), \\ a_1 &= b_1(p-1)p\varepsilon^2 + O(\varepsilon^4), \\ a_2 &= b_1(p^2 - 2p + 2)\varepsilon^2 + O(\varepsilon^4). \end{aligned}$$

Defining new coordinates and time by $x = \varepsilon^2 u$, $y = \varepsilon^3 v$, $\tau = \varepsilon t$, we get

$$\begin{cases} \dfrac{du}{d\tau} &= v \\ \dfrac{dv}{d\tau} &= \left(\dfrac{1}{p}b_2 - \dfrac{p-1}{2p^2}b_1^2\right) + \dfrac{p^4}{2(1-p)}\left(u - \dfrac{(p-1)^2}{p^3}b_1\right)^2 \\ & + \left[b_1(p^2 - 2p + 2)v - p^3 uv + \dfrac{1}{p}b_3\Gamma(\Omega\tau)\right]\varepsilon + O(\varepsilon^2), \end{cases} \tag{16}$$

where ε is a small parameter, and so, setting $z = u - \dfrac{(p-1)^2}{p^3}b_1$, we obtain

$$\begin{cases} \dfrac{dz}{d\tau} &= v \\ \dfrac{dv}{d\tau} &= (\dfrac{1}{p}b_2 - \dfrac{p-1}{2p^2}b_1^2) + \dfrac{p^4}{2(1-p)}z^2 + \\ & \varepsilon[b_1 v - p^3 zv + \frac{1}{p}b_3\Gamma(\Omega\tau)] + O(\varepsilon^2). \end{cases} \tag{17}$$

Let $q = \dfrac{p^4}{2(1-p)}z$, $w = \dfrac{p^4}{2(1-p)}v$, then the equations (17) become

$$\begin{cases} \dfrac{dq}{d\tau} &= w \\ \dfrac{dw}{d\tau} &= -c^2 + q^2 + \varepsilon\left[b_1 w + \dfrac{(2(p-1)}{p}qw + \dfrac{p^3}{2(1-p)}b_3\Gamma(\Omega\tau)\right] \\ & +O(\varepsilon^2), \end{cases} \tag{18}$$

where

$$c^2 = \left[\frac{p}{2(p-1)}b_2 - \frac{b_1^2}{4}\right]p^2 \tag{19}$$

which is a strictly positive order one constant by assumption (4). In the following we suppose that $c > 0$.

The equations (18), with $b_3 = 0$, is a standard normal form for the Takens-Bogdanov bifurcation. Assume that $b_1 > 0$, $b_3 = 0$ and ε is a sufficiently small positive constant, then there are two stationary points, $S_\pm = (\pm c, 0)$, S_+ is a saddle (see [5]).

3 Chaos in the SIR model with periodic and almost periodic transmission rates

We can apply Melnikov method to the SIR model in the form (18) for sufficiently small ε. The unperturbed equation for $\varepsilon = 0$ is

$$\begin{cases} \dot{q} &= w \\ \dot{w} &= -c^2 + q^2, \end{cases} \tag{20}$$

where dots now denote differentiation with respect to the rescaled time τ and $c > O$. By integration we obtain the associated Hamiltonian function for this system:

$$H(q, w) = \frac{1}{2}w^2 + c^2 q - \frac{1}{3}q^3 \tag{21}$$

and so solutions lie on curves of constant H. Equation (20) has two stationary points $(c, 0)$, which is a saddle point, and $(-c, 0)$, which is a center. The center is surrounded by a continuous family of periodic orbits $(q_h(\tau), w_h(\tau))$, and these are bounded by a homoclinic orbit $(q_h(\tau), w_h(\tau))$ such that

$$\lim_{\tau \to \pm\infty} (q_h(\tau), w_h(\tau)) = (c, 0).$$

The Melnikov function for this problem is

$$M(\tau_0) = \tag{22}$$
$$\int_{-\infty}^{+\infty} w_h(\tau)\left[b_1 w_h(\tau) + \frac{2(p-1)}{p}q_h(\tau)w_h(\tau) + \frac{p^3}{2(1-p)}b_3\Gamma(\Omega(\tau+\tau_0))\right]d\tau.$$

The explicit form of the homoclinic solution [3, 5, 7] is

$$\begin{cases} q_h(\tau) &= c - 3c\, sech^2\left(\tau\sqrt{\frac{c}{2}}\right) \\ w_h(\tau) &= 3c\sqrt{2c}\, sech^2\left(\tau\sqrt{\frac{2}{c}}\right)\tan h\left(\tau\sqrt{\frac{c}{2}}\right). \end{cases}$$

Substituting these expressions into the integral (22), we obtain

$$\begin{aligned} M(\tau_0) &= \frac{6}{5}(2c)^{\frac{5}{2}}b_1 - \frac{6(p-1)}{7p}(2c)^{\frac{7}{2}} + \frac{3\pi\Omega^2 p^3 b_3}{(1-p)sinh\left(\frac{\Omega\pi}{(2c)^{\frac{1}{2}}}\right)}cos(\Omega\tau_0) \\ &+ \frac{p^3 b_3}{2(1-p)}\int_{-\infty}^{+\infty} w_h(\tau) sin\sqrt{2}\Omega(\tau+\tau_0)d\tau \end{aligned}$$

which leads to

$$\begin{aligned} M(\tau_0) &= \frac{6}{5}(2c)^{\frac{5}{2}}b_1 - \frac{6(p-1)}{7p}(2c)^{\frac{7}{2}} \\ &+ \frac{3\pi\Omega^2 p^3 b_3}{(1-p)}\left[\frac{cos(\Omega\tau_0)}{sinh\left(\frac{\Omega\pi}{(2c)^{\frac{1}{2}}}\right)} + 2\frac{cos(\sqrt{2}\Omega\tau_0)}{sinh\left(\frac{\sqrt{2}\Omega\pi}{(2c)^{\frac{1}{2}}}\right)}\right] \end{aligned} \tag{23}$$

The criterion for the existence of a zero of this Melnikov function is obtained from $|cos(\Omega\tau_0)| < 1$ and $|cos(\sqrt{2}\Omega\tau_0)| < 1$. If $b_1 = 0$, it takes a particularly simple form; $|b_3| > b_c$ where

$$b_c \simeq \frac{2(p-1)^2(2c)^{\frac{7}{2}}}{7\pi\Omega^2 p^4}.\frac{sinh\left(\frac{\Omega\pi}{(2c)^{\frac{1}{2}}}\right)sinh\left(\frac{\sqrt{2}\Omega\pi}{(2c)^{\frac{1}{2}}}\right)}{2sinh\left(\frac{\Omega\pi}{(2c)^{\frac{1}{2}}}\right) + sinh\left(\frac{\sqrt{2}\Omega\pi}{(2c)^{\frac{1}{2}}}\right)}. \tag{24}$$

Note that the expression

$$2sinh\left(\frac{\Omega\pi}{(2c)^{\frac{1}{2}}}\right) + sinh\left(\frac{\sqrt{2}\Omega\pi}{(2c)^{\frac{1}{2}}}\right)$$

is always different from zero since $\Omega \neq 0$. Thus if $|b_3| > b_c$ the stable and unstable manifolds of the fixed point in the Poincaré map of perturbed systems (18) intersect transversely for sufficiently small ε, and there is chaotic phenomenon in the sense of Smale horseshoe for the perturbed systems (18); while if $|b_3| < b_c$ there is no intersection between the stable and unstable manifolds; at $b_3 = b_c$ there is a tangential intersection between the stable and unstable manifolds.

4 The Numerical Simulations of Chaos

Like in [5] we have shown mathematically that general almost periodically forced epidemiological models (1) can have chaotic motion by using Melnikov's

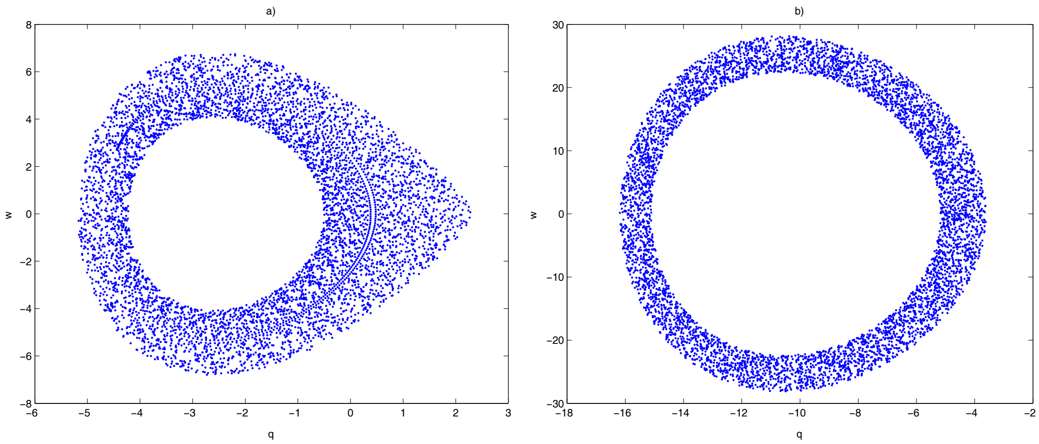

Figure 1.
The Poincaré map, in the almost periodic case, with $b_1 = 0$:

$a)\ b_2 = 1,\ b_3 = 5,\ p = 3,\ c = 2.5981,\ \Omega = 2,\ \varepsilon = 0.00002,\ q(0) = -5.19,\ w(0) = 0.$
$b)\ b_2 = 2,\ b_3 = 10,\ p = 4,\ c = 10.5410,\ \Omega = 2,\ \varepsilon = 0.00001,\ q(0) = -5.19,\ w(0) = 0.$

method close to a degenerate (Takens-Bogdanov) bifurcation point. We give now some numerical simulations for the model (1). The Poincaré maps of perturbed systems (18) is in figure 1 when we consider an almost periodic incidence rate. In this case the Poincaré maps of the perturbed systems (18) is a piece of points, so that we can affirm that the motion of system (18) is chaotic.

References

[1] R. M. Anderson, R. M. May, Infectious diseases of humans, Oxford: *Oxford University Press* (1991).

[2] J. L. Aron, I. B. Schwartz, Seasonality and period-doubling bifurcations in an epidemic model, *J. Theor. Biol.*, **110**, no 4, 665-679 (1984).

[3] N. C. Baesens, G. Nicolis, Complex bifurcations in a periodically forced normal form; *Z. Phys.* B **52**, 345-354 (1983)

[4] C. Corduneanu, *Almost Periodic Functions*, 2nd Edition, Chelsea-New York, (1989).

[5] O. Diallo and Y. Koné, Melnikov Analysis of Chaos in a General Epidemiological Model, *Nonlinear Analysis: Real World Applications*, Volume 8, no. 1, pp 20-26, (2007).

[6] P. Glendinning, L. P. Perry, Melnikov analysis of chaos in a simple epidemiological model; *J. Math. Biol.* **34**, 359-373 (1997).

[7] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields; *New York: Springer* (1983).

[8] H. W. Hethcote, P. Van Den Driessche, Some epidemiological models with nonlinear incidence, *J. Math. Biol.* **29**,271-287 (1991).

[9] Yu. A. Kuznetsov, C. Piccardi, Bifurcation analysis of periodic SEIR and SIR epidemic models, *J. Math. Biol.* **32**,109-121 (1994).

[10] W. M. Liu, S. A. Levin, Y. Iwasa, Influence of nonlinear incidence rates upon the behavior of simple SIR epidemiological models; *J. Math Biol.* **23**, 187-204 (1986).

[11] W. M. Liu, H. W. Hethcote, S. A. Levin, Dynamical behavior of epidemiological models with nonlinear incidence rate, *J. Math. Biol.* **25**, 359-380 (1987).

[12] L. F. Olsen, W. M. Shaffer, Chaos versus noisy periodicity: alternative hypotheses for childhood epidemics, *Science* **249**, 499-504 (1990).

[13] S. Wiggins, Global Bifurcations and Chaos, *New York: Springer* (1988).

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