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A Convergence Theorem via a New Fixed Point Theorem for Generalized Contractions

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Abstract

In this paper, we study the properties of τ^0-metrics introduced and studied by Du [1], and establish a new fixed point theorem and convergence theorem for τ^0-metrics.

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1. Introduction and preliminaries

Throughout this paper we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers, respectively. Let (X, d) be a metric space. We denote by $\mathcal{CB}(X)$ the class of all nonempty closed bounded subsets of X and $\mathcal{P}(X)$ the family of all nonempty subsets of X. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$, the distance between x and A. For any A, $B \in \mathcal{CB}(X)$, define a function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},$$

then $\mathcal{H}$ is said to be the *Hausdorff metric* on $\mathcal{CB}(X)$ induced by the metric d on X. Let $T : X \to \mathcal{P}(X)$ be a multivalued map. A point p in X is a fixed point of T if $p \in Tp$. The set of fixed points of T is denoted by $\mathcal{F}(T)$.

A multivalued map T from X into $\mathcal{CB}(X)$ is said to be *k-contractive* if there exists a nonnegative real number k with $k < 1$ such that $\mathcal{H}(Tx, Ty) \le kd(x, y)$ for all $x, y \in X$. A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* [4,5] if the following conditions hold:

(τ1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(τ2) If $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \le M$ for some $M = M(x) > 0$, then $p(x, y) \le M$;

(τ3) For any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

(τ4) For $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is known that any w-distance [2] is a τ-function; see [4, Remark 2.1].

Let $p : X \times X \to [0, \infty)$ be a τ-function. For each $x \in X$ and $A \subseteq X$, let $p(x, A) = \inf_{y\in A} p(x, y)$. If $A \in \mathcal{CB}(X)$, then the *(p, δ)-neighborhood* $A_{(p,\delta)}$ of A is defined by $A_{(p,\delta)} = \{x \in X : p(x, A) < \delta\}$ and the *δ-neighborhood* A_δ of A is defined by $A_\delta = \{x \in X : d(x, A) < \delta\}$ for each $\delta > 0$, respectively.

2. Generalized Hausdorff metrics

Very recently, Du [1] first introduce the concepts of τ^0-functions and τ^0-metrics as follows.

Definition 2.1. [1] Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a τ^0-function (resp. w^0-distance) if it is a τ-function (resp. w-distance) on X with $p(x, x) = 0$ for all $x \in X$.

Example. Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$ and $0 < a < b$. Define the function $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \max\{a(y - x), b(x - y)\}.$$

Then p is nonsymmetric and hence p is not a metric. It is easy to see that p is a τ^0-function.

The following two Lemmas are needed.

Lemma 2.1. [1,4,5] Let (X, d) be a metric space and $p : X \times X \to [0, \infty)$ be any function. If p satisfies (τ3) and there exists a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2. **[1]** Let A be a closed subset of a metric space (X, d) and $p : X \times X \to [0, \infty)$ be any function. Suppose that p satisfies $(\tau 3)$ and there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0$ *if and only if* $u \in A$.

From Lemma 2.2 and the definition of τ^0-function, we obtain the following property.

Lemma 2.3. Let (X, d) be a metric space and p be a τ^0-function. For any A, $B \in \mathcal{CB}(X)$, define $\delta_p(A, B) = \sup_{x\in A} p(x, B)$. Then for A, B, $C \in \mathcal{CB}(X)$, the following hold:

(i) $\delta_p(A, B) = 0 \iff A \subseteq B$;

(ii) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B)$;

(iii) if $A \subseteq B$, then $\delta_p(A, C) \leq \delta_p(B, C)$;

(iv) if $B \subseteq C$, then $\delta_p(A, C) \leq \delta_p(A, B)$;

(v) $\delta_p(A, B \cap C) \geq \max\{\delta_p(A, B), \delta_p(A, C)\}$;

(vi) $\delta_p(A, B \cup C) \leq \min\{\delta_p(A, B), \delta_p(A, C)\}$;

(vii) $\delta_p(A \cap B, C) \leq \min\{\delta_p(A, C), \delta_p(B, C)\}$;

(viii) $\delta_p(A \cup B, C) = \max\{\delta_p(A, C), \delta_p(B, C)\}$.

Proof. We first prove (i). If $\delta_p(A, B) = 0$ then $p(a, B) = 0$ for all $a \in A$. By Lemma 2.2, we have $A \subseteq B$. Conversely, if $A \subseteq B$, then $\delta_p(A, B) = 0$ by Lemma 2.2 again. Therefore $\delta_p(A, B) = 0 \iff A \subseteq B$. The proofs of (ii), (iii) and (iv) are straightfoward. Using (iii) and (iv), it is easy to verify that the conclusions of (v), (vi) and (vii) hold. To show (viii), since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, using (iii), we have $\max\{\delta_p(A, C), \delta_p(B, C)\} \leq \delta_p(A \cup B, C)$. Conversely, given $\varepsilon > 0$, there exists $u \in A \cup B$ such that $\delta_p(A \cup B, C) < p(u, C) + \varepsilon$. Then $\delta_p(A \cup B, C) < \max\{\delta_p(A, C), \delta_p(B, C)\} + \varepsilon$. Since ε is arbitrary, we obtain $\delta_p(A \cup B, C) \leq \max\{\delta_p(A, C), \delta_p(B, C)\}$ and hence (viii) holds. □

Definition 2.2. **[1]** Let (X, d) be a metric space and p be a τ^0-function. For any A, $B \in \mathcal{CB}(X)$, define a function $D_p : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ by

$$D_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where $\delta_p(A, B) = \sup_{x\in A} p(x, B)$, then D_p is said to be the τ^0-*metric* on $\mathcal{CB}(X)$ induced by p.

Clearly, any Hausdorff metric is a τ^0-metric, but the reverse is not true.

The following theorem is one of the main results in [1].

Theorem 2.1. **[1]** Every τ^0-metric D_p defined as in Def. 2.2 is a metric on $\mathcal{CB}(X)$.

Theorem 2.2. Let (X, d) be a metric space and D_p be a τ^0-metric on $\mathcal{CB}(X)$ induced by a τ^0-function p. Then for any A, $B \in \mathcal{CB}(X)$,

$$\begin{aligned} D_p(A, B) &= \sup_{x \in X} |p(x, A) - p(x, B)| \\ &= \inf\{\delta > 0 : A \subseteq B_{(p,\delta)}, B \subseteq A_{(p,\delta)}\}. \end{aligned}$$

Proof. It is easy to verify that the conclusion follows from Lemma 2.2 and Lemma 2.3, so we omit the proof. □

Applying Lemma 2.3, we can easy prove the following theorem.

Theorem 2.3. Let (X, d) be a metric space and D_p be a τ^0-metric on $\mathcal{CB}(X)$ induced by a τ^0-function p. Then for any A, B, C, $D \in \mathcal{CB}(X)$, $D_p(A \cup B, C \cup D) \leq \max\{D_p(A, C), D_p(B, D)\}$.

3. A new fixed point theorem and convergence theorem

Definition 3.1. A multivalued map $T : X \to \mathcal{CB}(X)$ is said to be (τ^0, k)-*contractive* or a (τ^0, k)-*contraction* if there exists a constant $0 \leq k < 1$ such that $D_p(Tx, Ty) \leq kp(x, y)$ for all x, $y \in X$.

The following result is a new fixed point theorem and inequality in complete metric spaces.

Theorem 3.1. Let (X, d) be a complete metric space and D_p be a τ^0-metric on $\mathcal{CB}(X)$ induced by a τ^0-function p. For each $i \in \{1, 2\}$, let $T_i : X \to \mathcal{CB}(X)$ be a (τ^0, k)-contractive multivalued map. Then the following hold:

(i) $\mathcal{F}(T_i) \neq \emptyset$ for each $i \in \{1, 2\}$;

(ii) If $p(x, \cdot)$ is l.s.c. for all $x \in \mathcal{F}(T_i)$, $i \in \{1, 2\}$, then

$$D_p(\mathcal{F}(T_1), \mathcal{F}(T_2)) \leq \frac{1}{1-k} \sup_{x \in X} D_p(T_1(x), T_2(x)).$$

Proof. Without loss of generality, we only prove $\mathcal{F}(T_1) \neq \emptyset$. Let $\lambda \in (k, 1)$. Take $x_0 \in X$ and $x_1 \in T_1 x_0$. If $p(x_0, x_1) = 0$, since $p(x_0, x_0) = 0$, by ($\tau 4$), we have $x_0 = x_1 \in T_1 x_0$. So $x_0 \in \mathcal{F}(T_1)$. If $p(x_0, x_1) > 0$, since

$$p(x_1, T_1 x_1) \leq D_p(T_1 x_0, T_1 x_1) < \lambda p(x_0, x_1)$$

there exists $x_2 \in T_1 x_1$ such that $p(x_1, x_2) < \lambda p(x_0, x_1)$. If $p(x_1, x_2) = 0$, then $x_1 \in \mathcal{F}(T_1)$. Otherwise, there exists $x_3 \in T_1 x_2$ such that $p(x_2, x_3) < \lambda p(x_1, x_2)$. Continuing this process, we can obtain a sequence $\{x_n\}_{n=0}^{\infty}$ in X satisfying $x_n \in T_1(x_{n-1})$, $p(x_{n-1}, x_n) > 0$ and $p(x_n, x_{n+1}) < \lambda p(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Hence

$$p(x_n, x_{n+1}) < \lambda p(x_{n-1}, x_n) \ < \cdots < \lambda^n p(x_0, x_1)$$

for each $n \in \mathbb{N}$. We claim that $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$. Let $\alpha_n = \frac{\lambda^n}{1-\lambda} p(x_0, x_1)$, $n \in \mathbb{N}$. For any $m, n \in \mathbb{N}$ with $m > n$, we have

$$p(x_n, x_m) \leq \sum_{j=n}^{m-1} p(x_j, x_{j+1}) < \alpha_n. \tag{3.1}$$

Since $0 < \lambda < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and hence $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$. By Lemma 2.1, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $\xi \in X$ such that $x_n \to \xi$ as $n \to \infty$. From ($\tau 2$) and (3.1), we have

$$p(x_n, \xi) \leq \alpha_n \ \text{ for all } n \in \mathbb{N}.$$

Since $x_{n+1} \in T_1(x_n)$, $p(x_{n+1}, T_1 \xi) < \lambda p(x_n, \xi)$ for each $n \in \mathbb{N}$. So there exists $\{a_n\} \subset T_1 \xi$ such that $p(x_{n+1}, a_{n+1}) < \lambda p(x_n, \xi) \leq \alpha_n$ for each $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} p(x_n, a_n) = 0$. By ($\tau 3$), $\lim_{n\to\infty} d(x_n, a_n) = 0$. Since $d(a_n, \xi) \leq d(a_n, x_n) + d(x_n, \xi)$, we obtain $a_n \to \xi$ as $n \to \infty$. Since $T_1 \xi$ is closed and $\{a_n\} \subset T_1 \xi$, we have $\xi \in T_1 \xi$ or $\xi \in \mathcal{F}(T_1)$. Hence the conclusion (i) holds.

Next we verify (ii). Clearly, if $T_1 = T_2$, then we are done. So we assume that $T_1 \neq T_2$. Let $\gamma = \sup_{x \in X} D_p(T_1(x), T_2(x))$. Without loss of generality, we may assume that $\gamma < \infty$. Let $\varepsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} n k^n < \infty$, we can choose $c > 0$ such that $c \sum_{n=1}^{\infty} n k^n < 1$ and take $\varepsilon_0 = \frac{c}{1-k} \varepsilon$. Let $u_0 \in \mathcal{F}(T_1)$. Then $u_0 \in T_1 u_0$. Since

$$p(u_0, T_2 u_0) \leq D_p(T_1 u_0, T_2 u_0) \leq \gamma < \gamma + \varepsilon$$

there exists $u_1 \in T_2 u_0$ such that $p(u_0, u_1) < \gamma + \varepsilon$. By our assumption, $D_p(T_2 u_0, T_2 u_1) \leq k p(u_0, u_1)$. Since

$$p(u_1, T_2 u_1) \leq D_p(T_2 u_0, T_2 u_1) < k p(u_0, u_1) + k \varepsilon_0$$

there exists $u_2 \in T_2u_1$ such that $p(u_1, u_2) < kp(u_0, u_1) + k\varepsilon_0$. Continuing in this way, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in T_2(u_n)$ and $p(u_n, u_{n+1}) < kp(u_{n-1}, u_n) + k^n\varepsilon_0$ for each $n \in N$. It follows that

$$p(u_n, u_{n+1}) < kp(u_{n-1}, u_n) + k^n\varepsilon_0 < \cdots < k^n p(u_0, u_1) + nk^n\varepsilon_0$$

for each $n \in N$. Let $\beta_n = \frac{k^n}{1-k}p(u_0, u_1)$, $n \in \mathbb{N}$. For any m, $n \in N$ with $m > n$, we have

$$\begin{aligned} p(u_n, u_m) &\leq \sum_{j=n}^{m-1} p(u_j, u_{j+1}) \\ &< p(u_0, u_1)\sum_{j=n}^{\infty} k^j + \varepsilon_0 \sum_{j=n}^{\infty} jk^j \\ &< \beta_n + \frac{\varepsilon}{1-k} \end{aligned}$$

Since ε is arbitrary, we have

$$\sup\{p(u_n, u_m) : m > n\} \leq \beta_n \text{ for each } n \in \mathbb{N}. \tag{3.2}$$

Since $0 \leq k < 1$, $\lim_{n\to\infty} \sup\{p(u_n, u_m) : m > n\} = 0$. Applying Lemma 2.1 again, $\{u_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $\hat{v} \in X$ such that $u_n \to \hat{v}$ as $n \to \infty$. From (τ2) and (3.2), we have

$$p(u_n, \hat{v}) \leq \beta_n \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Since $u_{n+1} \in T_2(u_n)$, by (3.3), we have $p(u_{n+1}, T_2\hat{v}) \leq kp(u_n, \hat{v}) < \beta_n$ for each $n \in \mathbb{N}$. So there exists $\{w_n\} \subset T_2\hat{v}$ such that $p(u_{n+1}, w_{n+1}) < \beta_n$ for each $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} p(u_n, w_n) = 0$. By ($\tau$3), $\lim_{n\to\infty} d(u_n, w_n) = 0$. Since $d(w_n, \hat{v}) \leq d(w_n, u_n) + d(u_n, \hat{v})$, it follows $w_n \to \hat{v}$ as $n \to \infty$. By the closedness of $T_2\hat{v}$ and $\{w_n\} \subset T_2\hat{v}$, we obtain $\hat{v} \in \mathcal{F}(T_2)$. Furthermore, if $p(x, \cdot)$ is l.s.c. for all $x \in \mathcal{F}(T_i)$, $i \in \{1, 2\}$, then

$$\begin{aligned} p(u_0, \hat{v}) &\leq \liminf_{n\to\infty} p(u_0, u_n) \\ &\leq \liminf_{n\to\infty} \sum_{j=0}^{n-1} p(u_j, u_{j+1}) \\ &\leq p(u_0, u_1)\sum_{j=0}^{\infty} k^j + \varepsilon_0 \sum_{j=1}^{\infty} jk^j \\ &< \frac{1}{1-k}[p(u_0, u_1) + \varepsilon] \\ &< \frac{1}{1-k}(\gamma + 2\varepsilon). \end{aligned}$$

Since ε is arbitrary, we have $p(u_0, \hat{v}) \leq \frac{\gamma}{1-k}$. Note that $u_0 \in \mathcal{F}(T_1)$ is arbitrary, it follows that $\sup_{x\in\mathcal{F}(T_1)} p(x, \mathcal{F}(T_2)) \leq \frac{\gamma}{1-k}$. Similarly, for any $\zeta_0 \in \mathcal{F}(T_2)$, following the same argument, there exists $\hat{z} \in \mathcal{F}(T_1)$ such that $p(\zeta_0, \hat{z}) \leq \frac{\gamma}{1-k}$. Also it implies that $\sup_{x\in\mathcal{F}(T_2)} p(x, \mathcal{F}(T_1)) \leq \frac{\gamma}{1-k}$. Therefore we obtain that

$$\begin{aligned} D(\mathcal{F}(T_1), \mathcal{F}(T_2)) &= \max\{\sup_{x\in\mathcal{F}(T_1)} p(x, \mathcal{F}(T_2)), \sup_{x\in\mathcal{F}(T_2)} p(x, \mathcal{F}(T_1))\} \\ &\leq \frac{1}{1-k} \sup_{x\in X} D_p(T_1(x), T_2(x)). \end{aligned}$$

This completes the proof. □

Corollary 3.1. (Nadler [6]) Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a k-contraction. Then T has a fixed point in X.

The following conclusion is the well-known Banach contraction principle.

Corollary 3.2. [6] Let (X, d) be a complete metric space and $T : X \to X$ be a k-contraction. Then T has a unique fixed point in X.

Corollary 3.3. [3] Let (X, d) be a complete metric space. If T_1 and T_2 are k-contractive multivalued maps from X into $\mathcal{CB}(X)$, then

$$H(\mathcal{F}(T_1), \mathcal{F}(T_2)) \leq \frac{1}{1-k} \sup_{x\in X} H(T_1(x), T_2(x)).$$

Applying Theorem 3.1, we obtain the following convergence theorem.

Theorem 3.2. Let (X, d) be a complete metric space and D_p be a τ^0-metric on $\mathcal{CB}(X)$ induced by a τ^0-function p. For each $i \in \mathbb{N} \cup \{0\}$, suppose that T_i is a (τ^0, k)-contractive multivalued map from X into $\mathcal{CB}(X)$ such that $p(x, \cdot)$ is l.s.c. for all $x \in \mathcal{F}(T_i)$ and $\lim_{n\to\infty} D_p(T_n(x), T_0(x)) = 0$ uniformly for all $x \in X$. Then $\lim_{n\to\infty} D_p(\mathcal{F}(T_n), \mathcal{F}(T_0)) = 0$.

Proof. Given $\varepsilon > 0$. Since $\lim_{n\to\infty} D_p(T_n(x), T_0(x)) = 0$ uniformly for all $x \in X$, there exists $n_0 \in \mathbb{N}$ such that $\sup_{x\in X} D_p(T_n(x), T_0(x)) < (1-k)\varepsilon$ for all $n \geq n_0$. By Theorem 3.1,

$$D_p(\mathcal{F}(T_n), \mathcal{F}(T_0)) \leq \frac{1}{1-k} \sup_{x\in X} D_p(T_n(x), T_0(x)) < \varepsilon$$

for all $n \geq n_0$. Hence $\lim_{n\to\infty} D_p(\mathcal{F}(T_n), \mathcal{F}(T_0)) = 0$. □

Corollary 3.4. [3] Let (X, d) be a complete metric space and T_i be a k-contractive multivalued mappings from X into $\mathcal{CB}(X)$ for i =0, 1, 2, $\cdots$ such that $\lim_{n\to\infty} \mathcal{H}(T_n(x), T_0(x)) = 0$ uniformly for all $x \in X$. Then $\lim_{n\to\infty} \mathcal{H}(F(T_n), F(T_0)) = 0$.

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