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Three Positive Solutions for One-Dimensional p-Laplacian Boundary Value Problem with a Derivative Argument[1]

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Abstract

By applying a fixed point theorem due to Avery and Peterson, we study the existence of at least three positive solutions for a three point one-dimensional p-Laplacian boundary value problem. The interesting point is the nonlinear term f is involved with the first-order derivative explicitly.

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1. Introduction

In this paper, we will consider the positive solutions for the one-dimensional p-Laplacian differential equation,

$$(\phi_p(u'(t)))' + q(t)f(t, u(t), u'(t))) = 0, \qquad t \in (0, 1), \tag{1.1}$$

subject to the following boundary condition,

$$u(0) = 0, \qquad u(1) = u(\eta), \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$ with $p > 1$, is called a p-Laplacian operator. $\eta \in (0, 1)$ is a constant.

For (1.1), when the nonlinear term f does not depend on the first-order derivative, many authors study the equation,

$$(\phi_p(u'(t)))' + a(t)f(u) = 0, \quad t \in (0, 1). \tag{1.3}$$

One may see [1-6] and references therein. However, their works were done under the assumption that f is allowed to depend just on u, the first order derivative u' is not involved explicitly in the nonlinear term. More recently, Guo and Ge in [6] consider the existence of positive solutions for the following boundary value problem,

$$\begin{cases} x'' + f(t, x, x') = 0, & 0 < t < 1, \\ x(0) = 0, \quad u(1) = \alpha u(\eta). \end{cases} \tag{1.4}$$

They obtain the existence of at least one positive solution for (1.4).

So, motivated by all the works above, we are concerned with the existence of positive solutions for the problem (1.1), (1.2).

Throughout, it is assumed that:

$(H1)$: $f \in C([0, 1] \times [0, \infty) \times R,\ [0, \infty))$

$(H2)$: $q(t)$ is a nonnegative continuous function defined on $(0, 1)$, and $q(t) \not\equiv 0$ on any
subinterval of $(0, 1)$. In addition, $\int_0^1 q(t)dt < +\infty$.

2. Preliminary

In this section, we give some preliminaries and definitions.

Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provided that,
$\diamond$ $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$, and
$\diamond$ $u, -u \in P$ implies $u = 0$. □

Definition 2.2. The map α is said to be a nonnegative, continuous, concave functional on a cone P of a real Banach space E, if $\alpha : P \to [0, \infty)$ is continuous, and

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in P$ and $t \in [0, 1]$. Similarly, we say the map β is a nonnegative, continuous, convex functional on a cone P of a real Banach space E, if $\beta : P \to [0, \infty)$ is continuous, and

$$\beta(tu + (1-t)v) \leq t\,\beta(u) + (1-t)\,\beta(v),$$

for all $u, v \in P$ and $t \in [0, 1]$. □

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c and d, we define the following convex sets,

$$P(\gamma, d) = \{u \in P |\ \gamma(u) < d\},$$

$$P(\gamma, \alpha, b, d) = \{u \in P |\ b \leq \alpha(u), \gamma(u) \leq d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{u \in P |\ b \leq \alpha(u), \theta(u) \leq c, \gamma(u) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{u \in P |\ a \leq \psi(u), \gamma(u) \leq d\}.$$

To prove our main results, we need the following fixed point theorem due to Avery and Peterson in [7].

Theorem 2.1. Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d, $\alpha(u) \leq \psi(u)$ and $\|u\| \leq M\gamma(u)$, for all $u \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that,

$(S1)$: $\{u \in P(\gamma, \theta, \alpha, b, c, d) |\ \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for $u \in P(\gamma, \theta, \alpha, b, c, d)$;

$(S2)$: $\alpha(Tu) > b$, for $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > c$;

$(S3)$: $0 \notin R(\gamma, \psi, a, d)$, and $\psi(Tu) < a$ for $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$.

Then T has at least three fixed points, u_1, u_2, and $u_3 \in \overline{P(\gamma, d)}$, such that,

$$\gamma(u_i) \le d, \quad for\ i = 1, 2, 3, \qquad b < \alpha(u_1),$$
$$a < \psi(u_2), \quad with\ \ \alpha(u_2) < b \quad and \quad \psi(u_3) < a. \qquad \square$$

3. Existence of three positive solutions to (1.1), (1.2)

Let the Banach space $E = C^1[0,1]$ be endowed with the norm, $\|u\| := \max\limits_{0\le t\le 1} [u^2(t) + (u'(t))^2]^{\frac{1}{2}}$, and define the cone $P \subset E$ by $P = \{u \in E |\ u(t) \ge 0, u(0) = 0, u\ is\ concave\ on\ [0,1]\}$. It follows from $(H2)$ that there exists a natural number $k \ge \max\{\frac{1}{\eta}, \frac{2}{1-\eta}\}$ such that $0 < \int_{\frac{1}{k}}^{1-\frac{1}{k}} q(t)dt < +\infty$. For notational convenience, we denote,

$$\lambda = \phi_p^{-1}(\int_0^1 q(r)dr),$$
$$L = \min\{\int_{\frac{1}{k}}^{\frac{1+\eta}{2}} \phi_p^{-1}(\int_s^{\frac{1+\eta}{2}} q(r)dr)ds, \int_{\frac{1}{k}}^{\eta} \phi_p^{-1}(\int_s^{\eta} q(r)dr)ds \ + \int_{\frac{1+\eta}{2}}^{1-\frac{1}{k}} \phi_p^{-1}(\int_{\frac{1+\eta}{2}}^{s} q(r)dr)ds\},$$
$$Q = \max\{\int_0^{\frac{1+\eta}{2}} \phi_p^{-1}(\int_s^{\frac{1+\eta}{2}} q(r)dr)ds, \int_0^{\eta} \phi_p^{-1}(\int_s^{1} q(r)dr)ds + \int_{\frac{1+\eta}{2}}^{1} \phi_p^{-1}(\int_{\frac{1+\eta}{2}}^{s} q(r)dr)ds\}.$$

Define the nonnegative, continuous, concave functional α, the nonnegative continuous convex functional θ, γ, and the nonnegative, continuous functional ψ on the cone P by

$$\gamma(u) = \max_{0\le t\le 1} |u'(t)|, \quad \psi(u) = \theta(u) = \max_{0\le t\le 1} |u(t)|, \quad \alpha(u) = \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}} |u(t)|.$$

Then from the definition of $\|u\|$, we have $\|u\| \le \sqrt{2} \max\limits_{0\le t\le 1}\{\theta(u), \gamma(u)\}$. And with the concavity of u, for all $u \in P$, we have $\frac{1}{k}\theta(u) \le \alpha(u) \le \theta(u) = \psi(u)$.

The following lemma follows easily.

Lemma 3.1. For $u \in P$, $\|u\| \le \sqrt{2}\gamma(u)$. $\square$

Our main result is as follows.

Theorem 3.1. Let $0 < a < b \le \frac{d}{2k}$ be given, and suppose that f satisfies the following conditions,

(ℓ_1): $f(t, \omega, v) \le \phi_p(\frac{d}{\lambda})$ for $(t, \omega, v) \in [0,1] \times [0, \sqrt{2}d] \times [-d, d]$;
(ℓ_2): $f(t, \omega, v) > \phi_p(\frac{kb}{L})$ for $(t, \omega, v) \in [\frac{1}{k}, 1 - \frac{1}{k}] \times [b, kb] \times [-d, d]$;
(ℓ_3): $f(t, \omega, v) < \phi_p(\frac{a}{Q})$ for $(t, \omega, v) \in [0,1] \times [0, a] \times [-d, d]$;

Then the boundary value problem (1.1), (1.2) has at least three positive solutions u_1, u_2, u_3 such that,

$$\max_{0\le t\le 1}|u_i'(t)|\le d,\quad for\ i=1,2,3,\qquad b<\min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}|u_1(t)|,$$
$$a<\max_{0\le t\le 1}|u_2(t)|,\quad with\quad \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}|u_2(t)|<b,\quad and\quad \max_{0\le t\le 1}|u_3(t)|<a.$$

Proof: We define an operator $T:P\to E$ by

$$(Tu)(t)=$$

$$\begin{cases}\int_0^t\phi_p^{-1}(\int_s^\sigma q(r)f(r,u(r),u'(r))dr)ds, & 0\le t\le\sigma,\\ \int_0^\eta\phi_p^{-1}(\int_s^\sigma q(r)f(r,u(r),u'(r))dr)ds+\int_t^1\phi_p^{-1}(\int_\sigma^s q(r)f(r,u(r),u'(r))dr)ds, & \sigma\le t\le 1,\end{cases} \tag{3.1}$$

where $\sigma_u\in[\eta,1]$ is the solution of the equation,

$$\int_\eta^x\phi_p^{-1}(\int_s^x q(r)f(r,u(r),u'(r))dr)ds=\int_x^1\phi_p^{-1}(\int_x^s q(r)f(r,u(r),u'(r))dr)ds.$$

From the definition of T, we deduce that for each $u\in P$, there is $Tu\in C^1[0,1]$ is nonnegative and satisfies (1.2). Moreover, $(Tu)(\sigma)$ is the maximum value of Tu on $[0,1]$. It is easy to obtain that $(Tu)'$ is continuous and nonincreasing in $[0,1]$, then we have $Tu\in P$. Then, a standard argument shows that $T:P\to P$ is completely continuous, and each fixed point of T in P is a solution of (1.1), (1.2).

We now show that the conditions of Theorem 2.1 are satisfied. The proof is divided into the following four steps.

(1) : We will show that condition (ℓ_1) implies that, $T:\ \overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$. (3.2)

In fact, for $u\in\overline{P(\gamma,d)}$, there is $\gamma(u)=\max_{0\le t\le 1}|u'(t)|\le d$, we have,

$$\gamma(Tu)=\max_{0\le t\le 1}|(Tu)'(t)|=\max\{(Tu)'(0),-(Tu)'(1)\}$$
$$=\max\{\phi_p^{-1}(\int_0^\sigma q(r)f(r,u(r),u'(r))dr),\ \phi_p^{-1}(\int_\sigma^1 q(r)f(r,u(r),u'(r))dr)\}\le\frac{d}{\lambda}\lambda=d.$$

Thus, we have, $\gamma(Tu)\le d$. Then, (3.2) holds.

(2) : We show that the condition $(S1)$ in Theorem 2.1 holds.

Take $u(t)=-2kb(t^2-t)$, for $t\in[0,1]$. It is easy to see that $u(t)\in P(\gamma,\theta,\alpha,b,kb,d)$ and $\alpha(u)>b$. Hence, $\{u\in P(\gamma,\theta,\alpha,b,kb,d)\mid\alpha(u)>b\}\neq$

$\emptyset$. Thus, we have,

$$\alpha(Tu) = \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}} |(Tu)(t)| \ge \frac{1}{k}\max_{0\le t\le 1}|(Tu)(t)| = \frac{1}{k}Tu(\sigma)$$
$$\ge \frac{1}{k}\min\{\int_{\frac{1}{k}}^{\frac{1+\eta}{2}} \phi_p^{-1}(\int_s^{\frac{1+\eta}{2}} q(r)f(r,u(r),u'(r))dr)ds,$$
$$\int_{\frac{1}{k}}^{\eta} \phi_p^{-1}(\int_s^{\eta} q(r)f(r,u(r),u'(r))dr)ds + \int_{\frac{1+\eta}{2}}^{1-\frac{1}{k}} \phi_p^{-1}(\int_{\frac{1+\eta}{2}}^{s} q(r)f(r,u(r),u'(r))dr)ds\} > \frac{1}{k}\frac{kb}{L}L = b.$$

Hence, we obtain that, $\alpha(Tu) > b$, for all $u \in P(\gamma,\theta,\alpha,b,kb,d)$. Consequently, condition $(S1)$ in Theorem 2.1 holds.

(3) : We prove $(S2)$ of Theorem 2.1 is satisfied.

With $\theta(Tu) > kb$, for all $u \in P(\gamma,\alpha,b,d)$, we have, $\alpha(Tu) \ge \frac{1}{k}\theta(Tu) > \frac{1}{k}kb = b$. Thus, condition $(S2)$ in Theorem 2.1 satisfied.

(4) : Finally, we prove $(S3)$ of Theorem 2.1 is also satisfied. Obviously, as $\psi(0) = 0 < a$, so $0 \notin R(\gamma,\psi,a,d)$. With $\psi(u) = a$, for $u \in R(\gamma,\psi,a,d)$, we have,

$$\psi(Tu) = \max_{0\le t\le 1}|(Tu)(t)| = Tu(\sigma)$$
$$\le \max\{\int_0^{\frac{1+\eta}{2}} \phi_p^{-1}(\int_s^{\frac{1+\eta}{2}} q(r)f(r,u(r),u'(r))dr)ds,$$
$$\int_0^{\eta} \phi_p^{-1}(\int_s^{1} q(r)f(r,u(r),u'(r))dr)ds + \int_{\frac{1+\eta}{2}}^{1} \phi_p^{-1}(\int_{\frac{1+\eta}{2}}^{s} q(r)f(r,u(r),u'(r))dr)ds\}$$
$$\le \frac{a}{Q}\max\{\int_0^{\frac{1+\eta}{2}} \phi_p^{-1}(\int_s^{\frac{1+\eta}{2}} q(r)dr)ds, \int_0^{\eta} \phi_p^{-1}(\int_s^{1} q(r)dr)ds + \int_{\frac{1+\eta}{2}}^{1} \phi_p^{-1}(\int_{\frac{1+\eta}{2}}^{s} q(r)dr)ds\} = a.$$

Thus, condition $(S3)$ in Theorem 2.1 holds.

Then, Theorem 3.1 is proved by Theorem 2.1. □

4. Example

Example 4.1. Let $p = 3$ and $q(t) = 1$ in (1.1) and $\eta = \frac{1}{8}$ in (1.2), we consider the following boundary value problem,

$$\begin{aligned} &(|u'(t)|u'(t))' + f(t,u(t),u'(t)) = 0, \qquad 0 < t < 1,\\ &u(0) = 0, \qquad u(1) = u(\frac{1}{8}), \end{aligned} \tag{4.1}$$

where

$$f(t,\omega,v) = \begin{cases} t + 9600\ \omega^{10} + \frac{1}{20}(\frac{v}{5\times 10^{10}})^3, & \omega \le 16,\\ t + 9600\cdot 16^{10} + \frac{1}{20}(\frac{v}{5\times 10^{10}})^3, & \omega \ge 16. \end{cases}$$

Choose $a = \frac{2}{3}, b = 1, k = 16, d = 5 \times 10^{10}$, then it is easy to verify that the conditions of Theorem 3.1 hold. So, we have that boundary value problem (4.1) has at least three positive solutions u_1, u_2, u_3 such that,

$$\max_{0\le t\le 1} |u_i'(t)| \le 5 \times 10^{10}, \quad \textit{for } i = 1, 2, 3, \qquad 1 < \min_{\frac{1}{16}\le t\le \frac{15}{16}} |u_1(t)|,$$

$$\frac{2}{3} < \max_{0\le t\le 1} |u_2(t)|, \quad \textit{with} \quad \min_{\frac{1}{16}\le t\le \frac{15}{16}} |u_2(t)| < 1, \quad \textit{and} \quad \max_{0\le t\le 1} |u_3(t)| < \frac{2}{3}. \ \Box$$

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