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A Remark on Total Domination Critical Graphs

N. Jafari Rad and S. Rahimi Sharebaf

Department of Mathematics
Shahrood University of Technology, Shahrood, Iran
n.jafarirad@shahroodut.ac.ir

Abstract

A graph G with no isolated vertex is total domination vertex critical if for any vertex v of G that is not adjacent to a vertex of degree one, the total domination number of $G - v$ is less than the total domination number of G. We call these graphs γ_t-critical. In this paper, we disprove a conjecture posed in a recent paper(On an open problem concerning total domination critical graphs, Expo. Math. 25 (2007), 175-179).

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1 Introduction

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set*, if each vertex of G is dominated by some vertices of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set S is called a *total dominating set* if each vertex of G is dominated by some vertices of S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. An *efficient total dominating set* for a graph G is a set $S \subseteq V(G)$ for which the open neighborhoods $N(v), v \in S$, form a partition of $V(G)$. An *end-vertex* in a graph G is a vertex of degree one and a support vertex is one that is adjacent to an end-vertex. We call a total dominating set of cardinality $\gamma_t(G)$ a $\gamma_t(G)$-set. We also let $S(G)$ be the set of support vertices of G.

A vertex v in a graph G is a *domination critical* vertex, or just a critical vertex, if removing v decreases the domination number. A graph G is said to be *domination vertex critical* if any vertex of G is a critical vertex. There are many papers on this topic, and we refer the reader to [2].

Goddard, et. al. in [1] began the study of those graphs, which we call γ_t-critical, where the total domination number decreases on the removal of any vertex which is not a support vertex. A vertex v in a graph G is γ_t-*critical* if $\gamma_t(G-v) < \gamma_t(G)$. Since total domination is undefined for a graph with isolated vertices, a graph G is *total domination vertex critical*, or just γ_t-critical, if every vertex of G that is not adjacent to a vertex of degree one is γ_t-critical. In particular, if $\delta(G) \geq 2$, then G is γ_t-critical if every vertex of G is γ_t-critical. If G is γ_t-critical, and $\gamma_t(G) = k$, then G is called k-γ_t-*critical*. They posed the following open question:

Question 1: Characterize all γ_t-critical graphs of order $(\gamma_t(G)-1)\triangle(G)+1$.

The authors in [3] studied this question and proved some results on characterization of γ_t-critical graphs of order $(\gamma_t(G)-1)\triangle(G)+1$. There is the following result in [3].

Proposition 2: There is no $3-\gamma_t$-critical graph of order 9.

They posed the following conjecture:

Conjecture 3: For $r \geq 6$, there is no $3-\gamma_t$-critical graph of order $2r+1$.

In this paper, we first correct Proposition 2, and characterize all $3-\gamma_t$-critical graphs of order 9. We then disprove Conjecture 3, by introducing a $3-\gamma_t$-critical graph of order $2r+1$ for each $r \geq 6$. We use the following results:

Theorem 1.1, [3]: Any γ_t-critical graph G of order $n = \Delta(G)(\gamma_t(G)-1)+1$ is regular.

Theorem 1.2, [3]: The diameter of a 3-γ_t-critical r-regular graph of order $n = 2r+1$ is 2.

Theorem 1.3, [3]: Let G be a k-γ_t-critical r-regular graph of order $r(k-1)+1$, then k is odd, and r is even.

Theorem 1.4, [3]: Let G be a k-γ_t-critical r-regular graph of order $r(k-1)+1$ with k odd. Let $v \in V(G)$ and S be a $\gamma_t(G-v)$-set, then S is an efficient total dominating set for $G-v$.

Throughout this paper for a subset $S \subseteq V(G)$ we denote the subgraph of G induced by S, by $G[S]$. For a vertex v in a graph G we also use S_v for a $\gamma_t(G-v)$-set.

2 Main Results

In this section, we give the main results of the paper. We first correct Proposition 2, and then disprove Conjecture 3. Let F be the graph with vertex set $\{x, y, z, v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{xv_i : i = 1, 2, 3, 4\}\cup$
$\{v_2v_3, v_1v_5, v_1y, v_1v_6, v_2y, v_2v_6, v_3v_5, v_3z, v_4v_5, v_4z, v_4v_6, v_5y, yz, zv_6\}$.

It is easy to check that F is a $3-\gamma_t$-critical of order 9. We correct the content of Proposition 2, and characterize all $3-\gamma_t$-critical graphs of order 9.

Theorem 2.1 *A graph G of order 9 is $3-\gamma_t$-critical if and only if $G = F$.*

Proof. Let G be a $3-\gamma_t$-critical of order 9. It follows from Theorem 1.1, that G is 4-regular. Let $x \in V(G)$. We know by Theorem 1.2, that $diam(G) = 2$. Let V_i denote the set of all vertices of G at distance i from x, for $i = 1, 2$. So $|V_1| = |V_2| = 4$. It is obvious that $S_x \cap V_1 = \emptyset$, so $S_x \subseteq V_2$. Let $S_x = \{y, z\}$. Also let $V_1 = \{v_3, v_4, v_5, v_6\}$ and $V_2 = \{v_1, v_2, y, z\}$.

Claim 1. $|N(y) \cap V_1| = |N(z) \cap V_1| = 2$.

Proof. We prove that $|N(y) \cap V_1| = 2$. Assume to the contrary that $|N(y) \cap V_1| \neq 2$. Since G is 4-regular, we may assume that $\{v_1, v_2, v_3\} \subseteq N(y)$ and $\{v_4, v_5, v_6\} \subseteq N(z)$. For x to be totally dominated by S_z, it follows that $v_3 \in S_z$. There are the following possiblities.

1) If $x \in S_z$, then v_3 is adjacent to both v_1 and v_2. Then $z \in S_{v_3}$. But G is 4-regular, so there is a vertex $u \in \{v_4, v_5, v_6\}$ such that u dominates both v_1 and v_2. Now in order to S_u dominates $G - u$ it follows that y is adjacent to some vertices in $\{v_4, v_5, v_6\}$. This contradicts $deg(y) = 4$.

2) If $x \notin S_z$, we may assume that $v_1 \in S_z$. In order to S_z dominates the vertices in $\{v_2, v_4, v_5, v_6\}$, it is required that $\deg(v_3) + \deg(v_1) \geq 9$, contradicting that G is 4-regular. Similarly, $|N(z) \cap V_1| = 2$. $\Diamond$

So $|N(y) \cap V_1| = |N(z) \cap V_1| = 2$. Let $v_1 \in N(y) \cap (V_2 \backslash \{z\})$ and $v_2 \in N(z) \cap (V_2 \backslash \{y\})$. Let $\{v_3, v_4\} \subseteq N(y)$ and $\{v_5, v_6\} \subseteq N(z)$. Now $|S_{v_1}| = 2$ and it follows from Theorem 1.4, that $|S_{v_1} \cap V_1| = |S_{v_1} \cap V_2| = 1$ and we have the following cases :

Case 1. If $v_2 \in S_{v_1}$, then we may assume that $v_3 \in S_{v_1}$. Now there is a vertex $w \in \{v_4, v_5, v_6\}$ such that v_3 is adjacent to w, and v_2 adjacent to both vertices of $\{v_4, v_5, v_6\} \backslash \{w\}$. In each case v_1 will be adjacent to all of the vertices of $\{v_4, v_5, v_6\}$. We examined all possible cases for w and noticed that if $w = v_4$ then G is not γ_t-critical, and if $w \in \{v_5, v_6\}$ then G is isomorphic to F.

Case 2. If $z \in S_{v_1}$, then we may assume that $v_5 \in S_{v_1}$. Then $\{v_3, v_4\} \subseteq N(v_5)$. Now we consider S_{v_2}. By Case 1 we can assume that $y \in S_{v_2}$. It follows from Theorem 1.4, that $S_{v_2} \cap \{v_3, v_4\} \neq \emptyset$. Let $v_3 \in S_{v_2}$. Now v_3 is adjacent to v_6, and v_1 is adjacent to all of the vertices in $\{v_4, v_6, v_2\}$. This is a contradiction, since $\deg(v_2) = 4$.$\Diamond$

Now, we disprove Conjecture 3. By Theorem 1.3, if there is a $3-\gamma_t$-critical graph of order $2r + 1$, then r is even. In the following we show that for any even $r \geq 6$ there is a $3 - \gamma_t$-critical graph of order $2r + 1$.

Theorem 2.2 *For any even $r \geq 6$ there is a $3 - \gamma_t$-critical graph of order $2r + 1$.*

Proof. Let $r \geq 6$ be an even integer, and let $n = 2r+1$. Let G be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, and edge set $E = \{v_iv_{i+1}, v_iv_{i+3}, ..., v_iv_{i+r-1} : i = 1, 2, ..., n\}$, which the additions are calculated in modulo n.

Any two adjacent vertices of G dominate at most $2r$ vertices of G, so $\gamma_t(G) \geq 3$. On the other hand $\{v_1, v_2, v_3\}$ is a total dominating set for G. So, $\gamma_t(G) = 3$.

Now we prove that G is γ_t-critical. Let $x \in V$. Without loss of generality we suppose that $x = v_{r+2}$. Now, $\{v_1, v_2\}$ is a total dominating set for $G - x$. Hence, G is γ_t-critical.$\diamondsuit$

References

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