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A New Interior Point Method for Nonlinear Complementarity Problem

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Abstract. In this paper, we present a new method of interior point to solve a class of the nonlinear problem of complementarity inspired from a study introduced by Censor et al. This method is regarded as reduction from the variational inequalities problem to a particular case. Under less restrictive constraints, we are able to generate a sequence of nonnegative elements and we establish the global convergence of the proposed algorithm. The introduction of the functions of Bregman allows us to calculate at each iteration two scalars by solving a suitable system and a nonlinear equation for only one variable. Some preliminary numerical results indicate that this method is completely promising.

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1. Introduction

Let us consider the following nonlinear complementarity problem noted (NCP): Find $\overline{x} \in \mathbb{R}^n$ such that

$$\overline{x} \geq 0, \quad F(\overline{x}) \geq 0, \quad \overline{x}^T F(\overline{x}) = 0. \tag{1}$$

where F is a nonlinear continuous operator from $\mathbb{R}^n$ into itself.
The complementarity problem (linear and nonlinear) arises in many cases, the most obvious one is obtained by taking $F(x) = \nabla f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}$ differentiable function, in which case (1) is the first order Karush Kuhn Tuker conditions associated to the following problem $\begin{cases} \min \ f(x) \\ x \geq 0 \end{cases}$.
In the other hand, (NCP) is considered as special case of variational inequalities

problem (VIP). This last consists to find $\overline{x} \in C$ such that

$$\langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \quad \text{for all} \quad x \in C. \tag{2}$$

where C is a closed convex subset of $\mathbb{R}^n$. It is not difficult to see that (VIP) is reduced to (NCP) if we take $C = \mathbb{R}^n_+$.

(NCP) has received lot of attention due to its various applications in operational research, the economic equilibrium models and the engineering design of technology [10,13]. Several numerical methods to solve (NCP) are developed, see [9, 11].
Among them, a class of the iterative methods based on the theory of maximum monotone operators and variational inequalities problem.

In this paper, we carried out a detailed study of a new approach of interior point to solve (NCP). This one is a reduction of a method suggested in [4] for a class of (VIP) and it uses the Bregman functions to the particular case (NCP). The objective of the authors in [4] is reducing the assumptions of convergence as well as the calculation of projection used in the algorithm.

Throughout this paper, we assume that solutions sets of (NCP) and (VIP) denoted by $\mathcal{T}_1$, $\mathcal{T}_2$ respectively are nonempty.

We show that if F is continuous and paramonotone, the sequence generated by the algorithm converges to a solution of (NCP) if $\mathcal{T}_1$ is nonempty.

The paper is structured as follows. Some preliminaries of projections mapping and monotonicity definitions are provided in the first part of section 2. The second part is devoted for the most significant projection methods and related to our study that we present in the chronological order of the works. Section 3 treats primarily the new alternative. The key word in all that is the Bregman functions and their associated distance, which allow us to compute the scalars in question by more suitable procedure. In the next section, we establish the convergence analysis of the algorithm. Thus, this approach presents certain theoretical disadvantages for which we have brought some interesting contributions. Finally, some computational reported in section 5 to demonstrate the properties and the efficiency of the method.

2.1. Preliminaries

In this section, we summarize some basic properties and related definitions which be used in the following discussion.
$\mathbb{R}^n$ denotes the space of real n-dimensional vectors, and $\|x\| = \sqrt{x^T x}$ the Euclidian norm, where $x^T y = \langle x, y \rangle$ is the inner product of $x, y \in \mathbb{R}^n$.
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1...n\}$ is the nonnegative orthant and its interior is $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_i > 0, \ i = 1...n\}$. Let C a convex closed subset of $\mathbb{R}^n$, and P_C present the orthogonal projection of $\mathbb{R}^n$ on C.

The basic property for the projection operator is:

$$(x - P_C(x))^T (y - P_C(x)) \leq 0, \text{ for all } x \in \mathbb{R}^n \text{ and for all } y \in C$$

For any real constant $\alpha > 0$, it well known that (VIP) is equivalent to the following projection equation:

(3) $$x = P_C\left(x - \alpha F\left(x\right)\right)$$

Definitions. Let $F : \mathbb{R}^n \to \mathbb{R}^n$,

• F is said to be monotone if

$$\left\langle F\left(x\right) - F\left(y\right), x - y\right\rangle \geq 0, \quad \text{for all } x, y \in C.$$

• F is strongly monotone if there exists a constant $\mu > 0$, such that

$$\left\langle F\left(x\right) - F\left(y\right), x - y\right\rangle \geq \mu \left\| x - y \right\|, \text{ for all } x, y \in C.$$

• F is paramonotone if F is monotone and satisfies

$$\left\langle F\left(x\right) - F\left(y\right), x - y\right\rangle = 0 \Rightarrow F\left(x\right) = F\left(y\right).$$

• F is Lipschitz continuous if there exists $L > 0$ such that

$$\left\| F\left(x\right) - F\left(y\right) \right\| \leq L \left\| x - y \right\|, \text{ for all } x, y \in R^n.$$

2.2. Projection Method :

There is an important class of projection methods known by the extragradient algorithms to solve (VIP), and which are inspired basically from the optimization methods. The classical theory of these methods is insufficient and too restrictive by its assumptions (strong monotony, the continuity of Lipschitz . . .). We will present in what follows, the most famous projection methods and relative to the new proposed algorithm.

The basic iterative procedure for F strongly monotone and continuous of Lipschitz is given by:

(4) $$x^{k+1} = P_C\left(x^k - \alpha F\left(x^k\right)\right), \quad \alpha > 0$$

This procedure rises directly from the relation (4). The proof is based on the fact that for $\alpha \in \left]0, \frac{2\mu}{L^2}\right[$,(where μ, L are the constants of strong monotonicity and lipschitz continuity respectively) the mapping $P_C\left(I - \alpha F\right)$ is contraction. Then the generated sequence $\left\{x^k\right\}$ converges to $\overline{x}$ the fixed point of this mapping and which is the unique solution of (VIP), see [2] for more details.

If the assumption of strong monotonicity isn't satisfied, then the procedure fails to converge even for choice of α_k variable.

In order to overcome this difficulty, Korpolevich has proposed a theoretical improvement of the preceding method [15], where the required hypotheses are: F is Lipschitz continuous and not necessarily strongly monotone. The iterative formulae associated to this method is:

$$\left\{ \begin{array}{l} y^k = P_C\left(x^k - \alpha F\left(x^k\right)\right) \\ x^{k+1} = P_C\left(x^k - \alpha F\left(y^k\right)\right) \end{array} \right. \tag{5}$$

To establish the convergence of Korpolevich algorithm, we must show at first that for the values of $\alpha \in \left]0, \frac{1}{L}\right[$ the vector $x^k - \alpha F\left(y^k\right)$ is the projection of x^k onto a hyperplane which separates x^k from the solutions set $\mathcal{T}_2$. Then, the iteration x^{k+1} is obtained through an other projection onto C .
The same, if the assumption of Lipschitz continuity is also removed, the decreasing of $\left\{\left\|x^k - \overline{x}\right\|\right\}$, $\forall \overline{x} \in \mathcal{T}_2$ is not guaranteed and an adaptive choice of α_k by using procedure search is needed at each iteration.

While basing on this idea, a new version known as modified Korpolevich method was presented in [14], where F is only monotone and continuous. The iterative procedure is as follows:

$$\left\{ \begin{array}{l} y^k = P_C\left(x^k - \sigma_k F\left(x^k\right)\right) \\ x^{k+1} = P_C\left(x^k - \lambda_k F\left(y^k\right)\right) \end{array} \right. \tag{6}$$

For the parameter σ_k is determined by a procedure of multiplier approximation (backtracking procedure) in such way that the hyperplane H_k normal to $F\left(y^k\right)$, passes through y^k (thus we can write
$H_k = \left\{x \in \mathbb{R}^n / \left\langle F\left(y^k\right), x - y^k\right\rangle = 0\right\}$) , separates x^k from $\mathcal{T}_2$, and $\left(x^k - \lambda_k F\left(y^k\right)\right)$ is the projection of x^k into H_k. From the monotonicity of F
and $\left(x^k - \lambda_k F\left(y^k\right)\right) \in H_k$, we assure the separation of x^k from $\mathcal{T}_2$,
and we find $\lambda_k = \left\langle F\left(y^k\right), x^k - y^k\right\rangle / \left\|F\left(y^k\right)\right\|^2$.

Remark 1. The major problem of all these algorithms lies in the computation of the projections which depends directly of the structure of C.
For the case of the complementarity problem, it is easy to calculate the projection on $C = \mathbb{R}^n_+$ by using this formulae:

$$P_{\mathbb{R}^n_+}(x) = \left\{ \begin{array}{ll} x_i, & \text{if } x_i \geq 0 \\ 0, & \text{if } x_i < 0 \end{array} \right. , \text{ for } i = 1, ..., n.$$

The difficulties which are posed in our case are the strong convergence hypotheses of the algorithms, the determination of parameters (α_k, σ_k, and λ_k) and the influence of their choice on the numerical behavior of the algorithms.

We present in what follows the correspondent algorithm to the modified Korpolevich method.

Algorithm 1.
1. Initialization
$k = 0,\ x^0 \in \mathbb{R}^n_+,\ \varepsilon_1 \in \left]0, 1\right[$ and $\sigma_{\max} > 0$.

2. Iterative step

2.1. First optimality criterion: If $\left\|f\left(x^k\right)\right\| = 0$ then stop, $x^k \in \mathcal{T}_1$ .Otherwise continue.

Calculate $y^k = P_{\mathbb{R}_+^n}\left(x^k - \sigma_{\max} F\left(x^k\right)\right)$.

2.2. Second optimality criterion: If $\left\|y^k - x^k\right\| = 0$ then stop, $x^k \in \mathcal{T}_1$ .Otherwise continue.

2.3. Selection of σ_k:

If $\left\|f\left(y^k\right) - f\left(x^k\right)\right\| \leq \left\|y^k - x^k\right\|^2 / 2\left(\sigma_{\max}\right)^2 \left\|f\left(x^k\right)\right\|$, take $\sigma_k = \sigma_{\max}$.

Otherwise continue select $\sigma \in]0, \sigma_{\max}[$ such that:

$\varepsilon_1 \left\|y^k - x^k\right\|^2 / 2\left(\sigma_{\max}\right)^2 \left\|f\left(x^k\right)\right\| \leq F\left(P_{\mathbb{R}_+^n}\left(x^k - \sigma F\left(x^k\right)\right) - F\left(x^k\right)\right) \leq$
$\left\|y^k - x^k\right\|^2 / 2\left(\sigma_{\max}\right)^2 \left\|f\left(x^k\right)\right\|$.

Take $\sigma_k = \sigma$, and calculate $y^k = P_{\mathbb{R}_+^n}\left(x^k - \sigma_k F\left(x^k\right)\right)$.

2.4. Third optimality criterion: If $\left\|f\left(y^k\right)\right\| = 0$ then stop, $y^k \in \mathcal{T}_1$ .Otherwise continue.

Calculate $\lambda_k = \left\langle F\left(y^k\right), x^k - y^k\right\rangle / \left\|F\left(y^k\right)\right\|^2$, and $x^{k+1} = P_{\mathbb{R}_+^n}\left(x^k - \lambda_k F\left(y^k\right)\right)$.

$k = k + 1$, and return to 2.1.

3. Method of interior point

3.1 Basic elements

We present in this paper, a new interior point method which was introduced by Censor and al. The principle of this method is the same one of the modified Korpolevich method, but this time the projections are carried out on hyperplanes which are much simpler.

Bregman functions

It's a class of functions introduced by Bregman [3]. These functions and the associated distances were used in several interesting applications in convex optimization [5, 6, 7, 8].

Let S be an open and convex subset of $\mathbb{R}^n$ and $\overline{S}$ its closure. We consider both the following functions:

$D_g : \overline{S} \times S \to R^+$ such that $D_g(x, y) = g(x) - g(y) - \langle \nabla g(y), x - y\rangle$

g is said a Bregman function with zone S denoted $g \in B_S$, if the conditions below hold

$\mathbf{B}_1$. g is continuous and strictly convex on $\overline{S}$.

$\mathbf{B}_2$. $g \in C^2(S)$.

$\mathbf{B}_3$. $\forall \alpha \in R$, the level sets $L_1(y, \alpha) = \left\{x \in \overline{S}\ D_g(x, y) \leq \alpha\right\}$ and $L_2 = \{y \in S\ / D_g(x, y) \leq \alpha\}$ are bounded, for all $y \in S$,and for all $x \in \overline{S}$ respectively.

$\mathbf{B}_4$. If $\left\{y^k\right\} \subset S$ converges to y^*,then $\lim\limits_{k\to\infty} D_g\left(y^*, y^k\right) = 0$.

$\mathbf{B}_5$. If $\left\{x^k\right\} \subset \overline{S}$ and $\left\{y^k\right\} \subset S$ are two sequences such that $\left\{x^k\right\}$ is bounded, $\lim\limits_{k\to\infty} y^k = y^*$, $\lim\limits_{k\to\infty} D_g\left(x^k, y^k\right) = 0$, then $\lim\limits_{k\to\infty} x^k = y^*$.

Bregman distance

Let g be a Bregman function with zone S, the application given previously, defines a generalized distance and it has the following properties:

P$_1$. $\forall x \in \overline{S}, \forall\ y \in S,\ D_g(x,y) \geqslant 0$.

P$_2$. $D_g(x,y) = 0 \Leftrightarrow x = y$.

P$_3$. $\forall p \in \overline{S}, \forall\ q,z \in S$ we have: $D_g(p,q) + D_g(q,z) - D_g(p,z) = \langle \nabla g(z) - \nabla g(q), p - q \rangle$.

For reasons of convergence, we need an additional condition on g :

B$_6$. $\forall\ u \in R^n, \exists\ x \in S$ such that $\nabla g(x) = u$.

Lemma 1. Under the conditions **B**$_1$ and **B**$_2$,we have this equivalence **B**$_6 \Leftrightarrow$ **B**$_7$ and **B**$_8$ where

B$_7$. If $\{x^k\}$ is sequence which converges to $x \in \partial S$ (boundary of S), then : $\langle \nabla g(x^k), u - x^k \rangle = +\infty, \forall\ u \in R^n \Leftrightarrow \lim_{k\to\infty} D_g(u, x^k) = +\infty$.

B$_8$. The level sets of the function ω are bounded, where $\omega : S \to R^+$ and $\omega(x) = \|\nabla g(x)\|$.

Bregman projection

Let $H = \{x \in R^n / a^t x = \beta\}$ where $0 \neq a \in \mathbb{R}^n$, $\beta \in R$, an hyperplane of $\mathbb{R}^n$, and $g \in B_S$ satisfying **B**$_7$ with $H \cap S \neq \varnothing$. Take $y \in S$, and consider the following problem:

$$(7) \qquad \begin{cases} \min D_g(x,y) \\ x \in H \cap \overline{S} \end{cases}$$

The problem (7) admit a unique solution $\overline{x}$ which presents the Bregman projection of y on H noted $\overline{x} = P_{BH}(y)$.

Moreover, the the solution is completely characterized by using the necessary and sufficient conditions of Karuch Kuhn Tuker

It exists $\lambda \in \mathbb{R}$ such that:

$$(8) \qquad \begin{cases} \nabla g(\overline{x}) = \nabla g(y) - \lambda\ a \\ \langle a, \overline{x} \rangle = \beta \end{cases}$$

There exist several Bregman functions for $C = \mathbb{R}^n_+$, the most important in the literature are:

1) $g(x) = \sum_{i=1}^n x_i \ln(x_i)$ the entropy function with the convention such that $0 \ln 0 = 0$.

2) $g(x) = \sum_{i=1}^n x_i \ln(x_i) - x_i$.

3) $g(x) = \sum_{i=1}^n (x_i^\mu - x_i^\sigma)$, with $\mu > 1$ and $\sigma \in]0,1[$

We have use the first function because it's least expansive than the other in practice.

Consequently, $D_g(x,y) = \sum_{i=1}^n x_i \ln\left(\frac{x_i}{y_i}\right) + y_i - x_i$,

$[\nabla g(x)]_i = \ln(x_i) + 1,\ \ \forall i = 1...n$, and $\nabla^2 g(x) = Diag\left(\frac{1}{x_i}\right), \quad \forall i = 1...n$.

3.2. Presentation of the method

In this part, we discuss about the presentation of the new method [4]. It's

based on the following results which we summarize here:
Let $g \in B_S$ satisfying $\mathbf{B}_6$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ a continuous mapping , such that $\bar{S} = C$. We define the following functions:
$h : S \times R_+^* \to S$
$(x, \sigma) \mapsto h(x, \sigma) = y$, where y is a solution of the system below:

$$\nabla g(y) = \nabla g(x) - \sigma F(x) \tag{9}$$

$\varphi, \bar{\varphi} : S \times R_+^* \to R$ where

$$\varphi(x, \sigma) = \left(\frac{1}{\sigma}\right) \langle F(x), x - h(x, \sigma)\rangle \tag{10}$$

$$\overline{\varphi}(x, \sigma) = \left(\frac{1}{\sigma}\right) \langle F(h(x, \sigma)), x - h(x, \sigma)\rangle \tag{11}$$

and ϕ: $S \to R$ where

$$\phi(x) = F(x)^t \left[\nabla^2 g(x)\right]^{-1} F(x) \tag{12}$$

We have the following properties:
$i)$ Under the assumption $\mathbf{B}_6$, h is well defined, because in this case y exists, unique and belongs to S. Moreover h is continuous and $h(x, 0) = x$, for all $x \in S$.
$ii)$ The functions $\varphi, \overline{\varphi}$ are continuous onto $S \times \mathbb{R}_+^*$, and for $\overline{x} \in S$, we have
$\lim\limits_{(x,\sigma)\to(\overline{x},0)} \varphi(x, \sigma) = \lim\limits_{(x,\sigma)\to(\overline{x},0)} \overline{\varphi}(x, \sigma) = \phi(\overline{x}) \geq 0.$
Being given $x^k \in S$ such that : $F\left(x^k\right) \neq 0$, we obtain y^k, x^{k+1} by resolving the system below (13-15):

$$\nabla g\left(y^k\right) = \nabla g\left(x^k\right) - \sigma_k F\left(x^k\right) \Leftrightarrow y^k = \left(h\left(x^k\right), \sigma_k\right) \tag{13}$$

$$\nabla g\left(x^{k+1}\right) = \nabla g\left(x^k\right) - \lambda_k F\left(y^k\right) \tag{14}$$

$$\left\langle F\left(y^k\right), x^{k+1} - y^k\right\rangle = (1 - \alpha_k)\left\langle F\left(y^k\right), x^k - y^k\right\rangle \tag{15}$$

where $\alpha_k \in [\hat{\alpha}, 1]$ and $\hat{\alpha} \grave{>} 0$.

Remark 2. From (8), we remark that y^k is not only the Bregman projection of x^konto a hyperplane of normal vector $F\left(x^k\right)$ supposed different from 0. The computation of λ_k is simple , owing to fact that we have informations concerning the hyperplane H_k. It passes through y^k, separates x^k from $\mathcal{T}_2$ and x^{k+1} is Bregman projection of x^k onto H_k. Consequently, H_k is explicitly given by this equation $\left\langle F\left(y^k\right), x - y^k\right\rangle = (1 - \alpha_k)\left\langle F\left(y^k\right), x^k - y^k\right\rangle$
We write the equations (12-14) in the case of complementarity problem as follows : (13) $\Leftrightarrow$ $y_i^k = x_i^k \exp\left(-\sigma_k F_i\left(x^k\right)\right)$
(14) $\Leftrightarrow$ $x_i^{k+1} = x_i^k \exp\left(-\lambda_k F_i\left(y^k\right)\right)$
(15) is nonlinear equation with λ_k as an unknown
$f(\lambda_k) = \sum_{i=1}^n x_i^k F_i\left(y^k\right) \exp\left(-\lambda_k F_i\left(y^k\right)\right) - (1 - \alpha_k) x_i^k F_i\left(y^k\right) - \alpha_k y_i^k F_i\left(y^k\right).$
The functions φ, $\overline{\varphi}$, and ψ become

$\varphi\left(x^k,\sigma_k\right)=\frac{1}{\sigma_k}\sum_{i=1}^n x_i^k F_i\left(x^k\right)\left(1-\exp\left(-\sigma_k F_i\left(x^k\right)\right)\right)$
$\overline{\varphi}\left(x^k,\sigma_k\right)=\frac{1}{\sigma_k}\sum_{i=1}^n x_i^k F_i\left(y^k\right)\left(1-\exp\left(-\sigma_k F_i\left(x^k\right)\right)\right).$
$\psi\left(x^k\right)=\sum_{i=1}^n x_i^k\left(F_i\left(x^k\right)\right)^2$

Optimality criteria
Proposition 1.
1) If $F\left(\overline{x}\right)=0\Rightarrow\overline{x}\in\mathcal{T}_2$
2) If $\overline{x}\in int\left(C\right)$ and $\overline{x}\in\mathcal{T}_2\Rightarrow F\left(\overline{x}\right)=0$. [12]

Unfortunately, the criterion given by 1) proposition 1, does not cover the possibility where the solution $\overline{x}\in\partial S$ and $F\left(x^k\right)\neq 0$. We have been able to regulate this problem as shows it the following proposition.

Proposition 2. Let $\overline{x}$ and $\overline{y}$ be a cluster points respectively for the sequences $\left\{x^k\right\}$and $\left\{y^k\right\}$ defined by the system (13 -15) given above.
If $F\left(\overline{x}\right)=F\left(\overline{y}\right)$, then $\overline{x}\in\mathcal{T}_2$.
We note that this condition hold under the paramonotonicity of F. We give the proof of this proposition at end of the analysis.

Now, we give the basic algorithm.
Algorithm 2.
1. Initialization
$k=0$, $x^0\in S,\varepsilon_1,\varepsilon_2>0$ such that: $\varepsilon_1+\varepsilon_2<1$.
2. **Iterative step:**
2.1. First optimality criterion: If $\left\|F\left(x^k\right)\right\|=0$ then stop, $x^k\in\mathcal{T}_2$.
Otherwise continue.
2.2. Selection of σ_k:
Choice $\sigma_{\max}>0$
Calculate $\varphi\left(x^k,\sigma_{\max}\right),\overline{\varphi}\left(x^k,\sigma_{\max}\right)$ and $\phi\left(x^k\right)$

(16) If $\overline{\varphi}\left(x^k,\sigma_{\max}\right)\geq\varepsilon_1\max\left\{\varphi\left(x^k,\sigma_{\max}\right),\phi\left(x^k\right)\right\}$

we take $\sigma_k=\sigma_{\max}$, otherwise select $\sigma\in\left]0,\sigma_{\max}\right[$ such that:

(17)
$$\varepsilon_1\max\left\{\varphi\left(x^k,\sigma\right),\phi\left(x^k\right)\right\}\leq\overline{\varphi}\left(x^k,\sigma\right)\leq\left(1-\varepsilon_2\right)\max\left\{\varphi\left(x^k,\sigma\right),\phi\left(x^k\right)\right\}$$

we take : $\sigma_k=\sigma$, then calculate : $y^k=h\left(x^k,\sigma_k\right)$
2.3. Second optimality criterion:
If $\left\|F\left(x^k\right)-F\left(y^k\right)\right\|=0$, then stop $x^k\in\mathcal{T}_2$. Otherwise continue
2.4. Bregman projection : solve the following system:
$$\begin{cases}\nabla g\left(x^{k+1}\right)=\nabla g\left(x^k\right)-\lambda_k F\left(y^k\right)\\ \left\langle F\left(y^k\right),x^{k+1}-y^k\right\rangle=\left(1-\alpha_k\right)\left\langle F\left(y^k\right),x^k-y^k\right\rangle\end{cases}$$
where the unknowns are x^{k+1},λ_k.
$k=k+1$, and return to 2.1.

4. Analysis convegence

Firstly we must show that it exists σ_k satisfying (17). We have $x^k\in S$,

$F\left(x^k\right) \neq 0$, from $\mathbf{B}_1$, $\mathbf{B}_2$ imply that the matrix $\nabla^2 g\left(x^k\right)$ is positive definite. Then, using (12) we obtain $\phi\left(x^k\right) \neq 0$.

Consider the following function: $\xi(\sigma) = \frac{\overline{\varphi}\left(x^k,\sigma\right)}{\max\left\{\varphi\left(x^k,\sigma\right),\phi\left(x^k\right)\right\}}$.

On one hand, by using the second case of σ_k selection we conclude that $\xi(\sigma_{\max}) < \varepsilon_1$, and from the properties of $\varphi, \overline{\varphi}$ imply $\lim\limits_{\sigma\to 0}\xi(\sigma) = 1$.
In the other hand, the continuity of ξ, $\xi(\sigma_{\max}) < 1$, and $\varepsilon_1 < 1-\varepsilon_2$ assure the existence of an interval $\left[\sigma', \sigma''\right] \subset]0,\ \sigma_{\max}[$ such that
$\forall \sigma \in \left[\sigma', \sigma''\right],\ \varepsilon_1 < \xi(\sigma) < (1-\varepsilon_2)$.

4.1. Selectionof σ_k

See [12]. Being given, the hyperplane H_k defined by (15).
If $F\left(x^k\right) \neq 0$, we have:

- $\left\langle F\left(y^k\right), z - y^k\right\rangle - (1-\alpha_k)\left\langle F\left(y^k\right), x^k - y^k\right\rangle \leq 0, \forall z \in \mathcal{T}_2$.
- $\left\langle F\left(y^k\right), x^k - y^k\right\rangle - (1-\alpha_k)\left\langle F\left(y^k\right), x^k - y^k\right\rangle = \alpha_k\left\langle F\left(y^k\right), x^k - y^k\right\rangle > 0$.

These last inequalities express the separation of x^k from $\mathcal{T}_2$, and they are satisfied under the assumption $\left\langle F\left(y^k\right), x^k - y^k\right\rangle > 0$ which depends directly from the selection of σ_k.

First criterion of selection

Let $\sigma_{\max} > 0$ be a initial value, and $F\left(x^k\right) \neq 0$. We have:

$$(16)\ \Leftrightarrow \left\langle F\left(y^k\right), x^k - y^k\right\rangle \geqslant \varepsilon_1 \max\left\{\left\langle F\left(x^k\right), x^k - y^k\right\rangle,\ \sigma_{\max}\left(F(x)^T\left[\nabla^2 g(x)\right]^{-1}F(x)\right)\right\}$$
$$\Leftrightarrow \left\langle F\left(y^k\right), x^k - y^k\right\rangle > 0.$$

Since $F\left(x^k\right) \neq 0$, $x^k \in S$ and the matrix $\nabla^2 g\left(x^k\right)$ is positive definite.
Then we take $\sigma_k = \sigma_{\max}$, if this criterion isn't verified, we must select an other $\sigma_k \in]0, \sigma_{\max}[$ that the existence is entirely assured.

Second criterion of selection

The second criterion is given by the double inequality (17) which guaranteed the separation and avoid the smallest values for σ_k.

Before given the principles stages of convergence, we note that the sequence $\left\{D_g\left(z, x^k\right)\right\}_k, \forall z \in \mathcal{T}_2$ is decreasing under the condition
$\left\langle \nabla g\left(x^k\right) - \nabla g\left(x^{k+1}\right), z - x^{k+1}\right\rangle \leq 0,\ \forall z \in \mathcal{T}_2$. See [12].

Lemma 2. If $\mathcal{T}_2 \neq \emptyset$, then $\left\langle \nabla g\left(x^k\right) - \nabla g\left(x^{k+1}\right), z - x^{k+1}\right\rangle \leq 0,\ \forall z \in \mathcal{T}_2$.

Lemma 3. Let F be a continuous and monotone mapping, then
$i)$ $z \in \mathcal{T}_2 \Leftrightarrow \forall x \in C, \langle F(x), x - z\rangle \geq 0$.
$ii)$ Further more if F is paramonotone, $z \in \mathcal{T}_2$ and $\overline{x} \in C$ satisfying $\langle F(\bar{x}), z - \bar{x}\rangle = 0$, then $\bar{x}$ is solution of (VIP).

Lemma 4. If the sequence $\left\{x^k\right\}$ is bounded, then $\left\{y^k\right\}$ is bounded.

Lemma 5. The assumption $\mathbf{B}_7$ holds if and only if $\lim\limits_{k\to\infty} D\left(u, x^k\right) = \infty, \forall u \in S$, and for all sequence $\left\{x^k\right\} \subset S$ which converges to a point $\overline{x} \in \partial S$.

For the demonstration of the algorithm convergence, it is enough to prove the

following results:
1. The sequence $\left\{x^k\right\}$ is well defined and belongs to S.
2. The sequence $\left\{x^k\right\}$ admits a limit $\overline{x} \in C$.
3. The limit $\overline{x}$ satisfies the optimality condition.

Indeed, for the first point the iteration y^k is well defined (from the definition of h). The fact that $H_k \cap S \neq \varnothing$ (because there exist at least one element $\hat{x}^k = y^k + (1-\alpha_k)\left(x^k - y^k\right)$ which belongs to H_k and S) where the possibility to applied the Bregman projection. Then x^k exists, unique and belongs to S.
The demonstration of second point is based on the following proposition which rises from lemma 2 and 3.

Proposition 3. If $\mathcal{T}_2 \neq \emptyset$, then:
i) $\left\{x^k\right\}$ is bounded.
ii) $\lim\limits_{k\to+\infty}\left(x^{k+1} - x^k\right) = 0$.
iii) If $\bar{x}$ is cluster point for the sequence $\left\{x^k\right\}$ belonging to $\mathcal{T}_2$,then $\lim\limits_{k\to+\infty} x^k = \bar{x}$.

Proof. *i*) By using the third property of Dg, taking $p = z$, $q = x^{k+1}$, $r = x^k$, and from lemma 1 we obtain

$$D_g\left(z, x^{k+1}\right) \leq D_g\left(z, x^k\right) - D_g\left(x^{k+1}, x^k\right) \tag{18}$$

Thus, the sequence $\left\{D_g\left(z, x^k\right)\right\}_k$, $\forall z \in \mathcal{T}_2$ is decreasing $\forall k \geqslant 0$.
From the condition $\mathbf{B}_3$, we have as consequence $\left\{x^k\right\}$ is bounded.
ii) The sequence $\left\{D_g\left(z, x^k\right)\right\}_k$, $\forall z \in \mathcal{T}_2$ is positive $\forall kS \geqslant 0$, and from (17) we have $0 \leq D_g\left(x^{k+1}, x^k\right) \leq D_g\left(z, x^k\right) - D_g\left(z, x^{k+1}\right)$.
Taking the limit $k \to +\infty$, imply $\lim\limits_{k\to+\infty} D_g\left(x^{k+1}, x^k\right) = 0$.
Using *i*) and the condition $\mathbf{B}_5$, we prove the result *ii*).
iii) Let $\{x^{j_k}\}$ be subsequence of $\left\{x^k\right\}$ such that $\lim\limits_{k\to+\infty} x^{j_k} = \bar{x}$ (hence $\left\{x^k\right\}$ is bounded from *i*)).
Then, from $\mathbf{B}_4$, we conclude that $\lim\limits_{k\to+\infty} D_g\left(\bar{x}, x^{j_k}\right) = 0$.
If $\bar{x} \in \mathcal{T}_2$, we have $\left\{D_g\left(\bar{x}, x^k\right)\right\}$ is a positive decreasing sequence, where it's convergent. Further more it admit a subsequence which converges to 0, then $\left\{D_g\left(\bar{x}, x^k\right)\right\}$ is also. Finally, the result $\lim\limits_{k\to+\infty} x^k = \bar{x}$ rises from $\mathbf{B}_5$. □

The third point of the analysis convergence passes through several propositions. Consider $\overline{x} \in \bar{S}$ such that:
1) $\bar{x} = \lim\limits_{k\to+\infty} x^{j_k}$, ($\{x^{j_k}\}$ is a subsequence of $\left\{x^k\right\}$, and let $\overline{x}$ be its limit).
The same for $\left\{y^k\right\}$, by using lemma 3, then
2) $\bar{y} = \lim\limits_{k\to+\infty} y^{j_k}$, 3) $\bar{\sigma} = \lim\limits_{k\to+\infty} \sigma_{j_k}$, 4) $\bar{\alpha} = \lim\limits_{k\to+\infty} \alpha_{j_k}$.
As consequence from proposition 3, we have 5) $\bar{x} = \lim\limits_{k\to+\infty} x^{j_k+1}$. □
The proposition below presents the case where the limit $\overline{x}$ satisfies the first

optimalty criterion (1), proposition 1).

Proposition 4.
$i)$ If there exists $\overline{x} \in \overline{S}$ satisfying 1) - 5), then $\langle F(\bar{y}), \bar{x} - \bar{y} \rangle = 0$.
$ii)$ If there exists $\overline{x} \in S$ satisfying 1) - 5), then $F(\bar{x}) = 0$.
Proof. We replace $\{x^k\}$ and $\{y^k\}$ by the subsequences $\{x^{j_k}\}$and $\{y^{j_k}\}$ in the equation of the hyperplane H_k, we find
$\langle F(y^{j_k}), x^{j_k+1} - y^{j_k} \rangle = (1 - \alpha_{j_k}) \langle F(y^{j_k}), x^{j_k} - y^{j_k} \rangle$.
Taking the limit $k \to +\infty$, and using the fact that F is continuous,we get
$\langle F(\bar{y}), \bar{x} - \bar{y} \rangle = (1 - \bar{\alpha}) \langle F(\bar{y}), \bar{x} - \bar{y} \rangle$.
In addition, we have $\bar{\alpha} \geq \hat{\alpha} > 0$, we find the result.
$ii)$ We distinguish two cases for the parameter $\overline{\sigma}$.

First case. ($\overline{\sigma}$ bounded away from 0)

Since $\overline{\sigma} > 0$, then we are in first selection criterion of σ_k which is express by the relation (15). But, we apply the relation to the subsequences $\{x^{j_k}\}$, $\{y^{j_k}\}$, and taking the limit $k \to +\infty$, we obtain
$0 \leq \left(F(\bar{x})^t [\nabla^2 g(\bar{x})]^{-1} F(\bar{x}) \right) \leq \frac{1}{\varepsilon_1 \sigma_{j_k}} \langle F(\bar{y}), \bar{x} - \bar{y} \rangle$.
Using the first result $i)$ and the fact that $\nabla^2 g(\bar{x})$ is positive definite, we conclude $\left(F(\bar{x})^t [\nabla^2 g(\bar{x})]^{-1} F(\bar{x}) \right) = 0 \Rightarrow F(\bar{x}) = 0$.

Second case. ($\overline{\sigma} = 0$, $\overline{\sigma}$ not bounded away from 0)

For this case, we use the second selection criterion of σ_k given by (16) and applied for $\{x^{j_k}\}$and $\{y^{j_k}\}$. Then, taking the limit $(x^{j_k}, \sigma_{j_k}) \to (\bar{x}, 0)$.
From the properties of φ and $\bar{\varphi}$, we obtain $\phi(\bar{x}) \leq (1 - \varepsilon_2) \phi(\bar{x})$,
where $F(\bar{x}) = 0$. □

The proposition 5 analyzes the case where $\overline{x} \in \partial S$, and $F(\bar{x}) \neq 0$.

Proposition 5. If there exists $\overline{x} \in \partial S$ satisfying 1) - 5), $F(\bar{x}) \neq 0$, and F is paramonotone, then $F(\overline{x}) = F(\overline{y})$.
Proof. The same, we replace $\{x^k\}$ and $\{y^k\}$ by the subsequences $\{x^{j_k}\}$and $\{y^{j_k}\}$ in the following relation $\varepsilon_1 \langle F(x^k), x^k - y^k \rangle \leq \langle F(y^k), x^k - y^k \rangle$, which rises from (13). Taking the limit $k \to +\infty$, we find $\langle F(\bar{x}), \bar{x} - \bar{y} \rangle \leq \frac{1}{\varepsilon_1} \langle F(\bar{y}), \bar{x} - \bar{y} \rangle$.
On one hand, $\langle F(\bar{x}), \bar{x} - \bar{y} \rangle \leq \frac{1}{\varepsilon_1} \langle F(\bar{y}), \bar{x} - \bar{y} \rangle = 0$.(proposition 4, i)).
On the other hand, $0 = \langle F(\bar{y}), \bar{x} - \bar{y} \rangle \leq \langle F(\bar{x}), \bar{x} - \bar{y} \rangle$ (F is monotone).
As consequence, $\langle F(\bar{x}) - F(\bar{y}), \bar{x} - \bar{y} \rangle = 0$

Using the paramonotoncity condition of F, we have the result $F(\bar{x}) = F(\bar{y})$.

□

Before presenting the main theorem in the convergence analysis, we give in brief the following propositions which enter in the framework of the proof of proposition 2.

Proposition 6.
$i)$ If the sequence $\{x^k\}$is bounded and has not cluster point in S,
then $\lim_{k \to +\infty} D_g(u, x^k) = +\infty$, $\forall u \in S$.

ii) In addition to the preceding condition and $\lim_{k \to +\infty} \left(x^{k+1} - x^k \right) = 0$, then $\forall u \in S$, there exists a cluster point $\overline{x}(u)$ of $\left\{ x^k \right\}$ belonging to ∂S such as $\langle F(\overline{x}(u)), u - \overline{x}(u) \rangle \geqslant 0$.

Proposition 7. If $\mathcal{T}_2 \neq \emptyset$, and $\left\{ x^k \right\}$ has no cluster point on S, then it exists $\overline{x}$ belonging to ∂S and $\overline{x}$ is (VIP) solution.

Proof. (**Proposition 2**)
Using the propositions 5 and 6 to prove that when the sequence $\left\{ x^k \right\}$ has no cluster point in S and all clusters point are in the boundary. Then for each $u \in S$ such that $\left\langle F\left(\overline{x}\left(u^k\right)\right), u^k - \overline{x}\left(u^k\right) \right\rangle \geqslant 0$. Finally, from proposition 7, we approach the solution z through points in S, we find a cluster point $\overline{x}$ such that $\langle F(\overline{x}), z - \overline{x} \rangle = 0$ and so $\overline{x}$ is (VIP) solution. □

Theorem 1. Let F be a continuous, paramonotone mapping and g a Bregman function satisfying $\mathbf{B}_6$ with zone $S = \mathbb{R}^n_{++}$. Then, the sequence generated by the preceding algorithm is well defined and converges if and only if $\mathcal{T}_1 \neq \emptyset$ and its limit $\overline{x} \in \mathcal{T}_1$.

Proof. The proof rises from all these propositions. For more details, see [4].

5.Numerical results

See [12]. We will present the whole of the obtaining results from the numerical implementation for the class of complementarity problem known by its importance. Let us note that for this class class, an adaptation of the basic algorithm is necessary. We also implemented the algorithm of modified korpolevich method, to compare the obtained results. The programs are carried out in Pascal- Turbo with precision of order $\varepsilon = 10^{-7}$and while taking $\varepsilon_1 = 0.3$, $\varepsilon_2 = 0.5$. The relaxation parameter is taken constant $\alpha_k = \alpha$, $\forall k \geqslant 0$. During the test, we have varied the formal parameters $\sigma_{\max}, \alpha$ and x^0 to see the influence of each one on the numerical behavior of the algorithm. We have proposed the Newton procedure to so solve the nonlinear equation of the unknown λ_k.

Example	Size	Iterations number		CPU (s)	
		Censor	Korpolevich m	Censor	Korpolevich m
1	2	169	736	0.05	0.06
2	3	31	8700 *	0.05	4.68
3	4	37	6127 *	0.06	2.14
4	4	75	84	0.06	0.06
5	6	200	732	0.06	0.11
6	9	8	22179 **	0.05	21.42
7	10	784	610	0.33	0.22
8	20	44	12117 **	0.05	62.37

(Table 1)

(*, ** mean that the precision is 10^{-4},10^{-3} respectively, for modified Korpolevich algorithm.

6. Conclusion

In general, the behavior of the interior point algorithm for the class (NCP) is definitely better than the modified Korpolevich algorithm. But, the execution time increased a little due to the Newton's procedure for the calculation of λ_k. We note the divergence of modified Korpolevich algorithm in the case where the solution $\overline{x} \in \partial S$ and $F(\overline{x}) \neq 0$ for the classes nonlinear (NCP). Through the tests carry out, we have noticed that the values of the parameters $\sigma_{\max}$, α vary in an interval $]0, 1]$. In particular, we can take for α values larger than the higher value of this theoretical interval and the results obtained in general are better. The principal contribution that we have brought for this study is to introduce the second criterion of optimality. Because if we are limited to use the first criterion and the only one given by Censor, then the case where the solution $\overline{x} \in \partial S$ and $F(\overline{x}) \neq 0$ will not be detected and the algorithm enter in an infinite cycle. The approach of interior point is considerably promising for the resolution of (VIP). We need to push this alternative to its higher improvement level.

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