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On Perron's Unit Root Tests in the Presence of an Innovation Variance Break

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Abstract

The unit root tests of Perron (1989, *Econometrica*) were designed to have power against the stationary alternative characterized by a break in the trend function. We show that all versions of Perron's (1989) tests can be over-sized when there is a break in the innovation variance. We propose modified Perron statistics based on the GLS transformation proposed by Kim, Leybourne, and Newbold (2002, *Journal of Econometrics*) that maintain size and have power against the trend-break stationary alternative. The modified Perron statistics weakens evidence against the unit root null for the Nelson-Plosser macroeconomic series.

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1. Introduction

Perron's (1989) seminal paper demonstrated that the conventional Dickey-Fuller (1979) test will spuriously accept the unit root null hypothesis if the trend-stationary alternative did not allow for a break in the trend function. Therefore, Perron (1989) developed unit root tests that are designed to have power against the trend-break stationary alternative.[1] Three different characterizations of the trend-break were considered, namely, (a) the Crash model that allows for a break in the intercept; (b) the Changing Growth model that allows for a break in the time trend with the two segments joined at the time of the break; and (c) the Mixed model that allows for a simultaneous break in the intercept and the time trend. The location of the break or break-date is assumed to be known to the practitioner. Perron's (1989) tests are based on a regression that nests the unit root

[1] We consider the Innovation Outlier model in which any change in the trend function evolves in the same manner as any other shock, see Perron (1989) for further details.

null and the appropriate trend-break stationary alternative hypothesis. The unit root test, denoted by t_P^i for i=A,B,C corresponding to the Crash, the Changing Growth, and the Mixed model respectively, is the t-test for the hypothesis that the coefficient on the first lag of the dependent variable is equal to one. Perron (1989) provides empirical evidence regarding the Nelson and Plosser (1982) macroeconomic series.

In a recent paper, Kim, Leybourne, and Newbold (2002) considered the problem of testing for the presence of a unit root when there is a break in the innovation variance.[2] [2] Given that Perron's (1989) statistics do not allow for a break in the innovation variance, we, in this paper, consider the limiting null behaviour of Perron's (1989) unit root tests in the presence of a break in the innovation variance. Our results indicate that the limiting null distributions of t_P^i (i=A,B,C) depend on both the break-fraction (τ^c), the pre-break variance (σ_1^2), and the post-break variance (σ_2^2). Simulation evidence reveals that the Perron statistics are over-sized if there is either a fall in the variance relatively early in the sample or an increase in the variance relatively late in the sample.

We adapt Kim, Leybourne, and Newbold's (2002) modified GLS strategy to correct for the size distortions in t_P^i (i=A,B,C). The limiting null distribution of our modified Perron unit root statistics, denoted by t_P^{i*} (i=A,B,C), in the presence of an innovation variance break are the same as those described by Perron (1989).[3] [3] Our simulation evidence confirms that the modified Perron statistics, t_P^{i*} (i=A,B,C), maintain their size in finite samples.

Finally, we illustrate the use of the modified Perron statistics by testing for the presence of a unit root in the thirteen macroeconomic series contained in the Nelson and Plosser (1982) data set. Our evidence shows that, for most series, there can be substantial changes in the innovation variance across the pre-break and post-break sub-samples. Given that Perron's (1989) statistics can be over-sized in the presence of a break in the innovation variance, we re-evaluate the empirical evidence by formally incorporating this potential break in the innovation variance. Our empirical results weaken the evidence against the unit root null hypothesis considerably. Specifically, while Perron (1989) rejected the unit root null hypothesis for ten series, we are only able to reject the unit root null for four series: Real GNP, Nominal GNP, Industrial Production, and Employment. Our analysis, therefore, suggests that the practitioner should use the modified Perron statistics if a break in the innovation variance is suspected.

The remainder of the paper is organized as follows. In Section 2, we discuss the data generating process, the null and alternative hypothesis, Perron's (1989) unit root tests, and their limiting null behaviour in the presence of a break in the innovation variance.

[2] Kim, Leybourne, and Newbold (2002) show that the conventional Dickey-Fuller (1979) pseudo t-statistics suffer from severe size distortions.

[3] For further details on the limiting distribution of the Perron (1989) statistics, see his Theorem 2 on pp.1373-1375.

In Section 3, we discuss modifications of the Perron's (1989) tests that incorporate the innovation variance break. We illustrate the use of the suggested modified Perron's (1989) tests within the context of the Nelson-Plosser series in Section 4. Section 5 contains some concluding remarks, and all proofs are relegated to an appendix.

2. Behaviour of Perron's Unit Root Statistics

Perron (1989) developed unit root tests that are designed to have power against a trend-break stationary alternative. The location of the break is assumed to be known a priori. For the asymptotic results, we assume that the break-date is a constant fraction of the sample size, that is, $T_b^c = \tau^c T$ with the break-fraction $\tau^c \in$ (0,1). Perron (1989) considers the following three different characterizations of the break under the stationary alternative:

Model (A): $$y_t = \mu_0 + \mu_1 DU_t^c + \gamma D_t^c + \mu_2 t + \rho y_{t-1} + \epsilon_t \quad (1)$$

Model (B): $$y_t = \mu_0 + \mu_2 t + \mu_3 DT_t^{*c} + \rho y_{t-1} + \epsilon_t \quad (2)$$

Model (C): $$y_t = \mu_0 + \mu_1 DU_t^c + \gamma D_t^c + \mu_2 t + \mu_3 DT_t^c + \rho y_{t-1} + \epsilon_t \quad (3)$$

where $DU_t^c = 1_{(t > T_b^c)}$ is an intercept dummy, $D_t^c = 1_{(t = T_b^c + 1)}$, $DT_t^{*c} = (t - T_b^c)\, 1_{(t > T_b^c)}$, $DT_t^c = t\, 1_{(t > T_b^c)}$, and ϵ_t is the error term. Model (A) or the Crash model includes the intercept dummy (DU_t^c) to allow for a break in the intercept, Model (B) or the Changing Growth model includes the trend dummy (DT_t^{*c}) to allow for a break in the time trend with the two segments joined at T_b^c, and Model (C) or the Mixed model includes an intercept dummy (DU_t^c) and a trend dummy (DT_t^c) to allow for a simultaneous break in the intercept and the time trend. In the general case, 'k' additional regressors $\{\Delta y_{t-j}\}_{j=1}^k$ are included in regressions (1)-(3) to eliminate correlation in the disturbance term. Typically, the value of the lag-truncation parameter (k) is unknown, and so a data-dependent method for choosing the appropriate value of k is used. For example, one may use Perron and Vogelsang's (1992) data-dependent method $k(t - sig)$ for selecting the lag-truncation parameter which is described in what follows. Specify an upper bound '$kmax$' for the lag-truncation parameter. The chosen value of the lag-truncation parameter (k^*) is determined according to the following 'general to specific' procedure: the last lag in an autoregression of order k^* is significant, but the last lag in an autoregression of order greater than k^* is insignificant. The significance of the coefficient is assessed using the 10% critical values based on a standard normal distribution.

Under the null hypothesis, the data generating process contains a unit root, that is:

$$y_t = \mu_0 + \gamma D_t^c + y_{t-1} + \epsilon_t \quad (4)$$

Perron (1989) presents the limiting null distribution of the unit root tests based on regressions (1)-(3), denoted by t_P^i for i=A,B,C corresponding to the Crash, the Changing Growth, and the Mixed model respectively, see Theorem 2 in Perron (1989). The limiting distribution of t_P^i (i=A,B,C) are non-standard and depend on the break-date. Therefore, Perron (1989) tabulates the critical values for a selection of break-dates to facilitate their use in empirical applications, see Tables IV.B, V.B, and VI.B on pp. 1376-1377 in Perron (1989).

We study the behaviour of t_P^i (i=A,B,C) in the presence of a break in the innovation variance (which may occur under the unit root null hypothesis or under the trend-break stationary alternative). We assume that the break in the innovation variance coincides with the break in the trend-function (if it exists). Following Kim, Leybourne, and Newbold (2002), we model the break in the innovation variance as:

$$\epsilon_t = \sigma_t \, \eta_t \tag{5}$$

with $\sigma_t^2 = \sigma_1^2 \, 1_{[t \leq \tau^c T]} + \sigma_2^2 \, 1_{[t > \tau^c T]}$, and η_t is a martingale difference sequence with $E(\eta_t^2 | \eta_{t-1}, ...) = 1$ and $E(|\eta_t|^{4+\beta} | \eta_{t-1}, ...) = \kappa < \infty$ for some $\beta > 0$. So, the innovation variance in the pre-break sample is σ_1^2 and that in the post-break sample is σ_2^2.

The following results characterize the limiting null distribution of t_P^i (i=A,B,C) when there is a break in the innovation variance. Let $B_1 = \int_0^{\tau^c} W(r)\, dr$, $B_2 = \int_{\tau^c}^1 W(r)\, dr$, $B_3 = \int_0^{\tau^c} r\, W(r)\, dr$, $B_4 = \int_{\tau^c}^1 r\, W(r)\, dr$, $B_5 = \int_0^{\tau^c} W(r)^2\, dr$, and $B_6 = \int_{\tau^c}^1 W(r)^2\, dr$.

Theorem 1: Suppose the true data generating process for the time series $\{y_t\}_{t=0}^T$ is given by equations (4) and (5). The limiting distribution of t-statistic for $H_o : \rho = 1$, denoted by t_P^A, based on the OLS regression (1) is given by:

$$t_P^A \Rightarrow \frac{1}{\sqrt{\{\sigma_1^2\, \tau^c + \sigma_2^2\, (1 - \tau^c)\}}} \; \frac{\left(V_A^{-1}\, W_A\right)_{[2,1]}}{\sqrt{\left(V_A^{-1}\right)_{[2,2]}}}$$

where V_A is a (2×2) symmetric matrix, and W_A is a (2×1) matrix given by:

$$\begin{aligned}
V_A[1,1] &= \frac{1}{12}\left[1 - 3\,\tau^c + 3\,(\tau^c)^2\right] \\
V_A[1,2] &= \sigma_1\, B_3 + \sigma_2\, B_4 - \frac{1}{2}\,\tau^c\,\sigma_1\, B_1 - \frac{1}{2}\,(1+\tau^c)\sigma_2\, B_2 \\
V_A[2,2] &= \sigma_1^2\left[B_5 - (\tau^c)^{-1}\, B_1^2\right] + \sigma_2^2\left[B_6 - (1-\tau^c)^{-1}\, B_2^2\right]
\end{aligned}$$

$$\begin{aligned}
W_A[1,1] &= \frac{1}{2}\,\sigma_1\,\tau^c\, W(\tau^c) + \frac{1}{2}\,\sigma_2\,(1-\tau^c)\,\{W(1) - W(\tau^c)\} - \sigma_1\, B_1 - \sigma_2\, B_2 \\
W_A[2,1] &= \frac{1}{2}\,\sigma_1^2\left\{W(\tau^c)^2 - \tau^c\right\} - (\tau^c)^{-1}\,\sigma_1^2\, W(\tau^c)\, B_1 \\
&\quad + \frac{1}{2}\,\sigma_2^2\left\{W(1)^2 - W(\tau^c)^2 - (1-\tau^c)\right\} - (1-\tau^c)^{-1}\,\sigma_2^2\,\{W(1) - W(\tau^c)\}\, B_2
\end{aligned}$$

Theorem 2: Suppose the true data generating process for the time series $\{y_t\}_{t=0}^{T}$ is given by equations (4) and (5). The limiting distribution of t-statistic for $H_o : \rho = 1$, denoted by t_P^B, based on the OLS regression (2) is given by:

$$t_P^B \Rightarrow \frac{1}{\sqrt{\{\sigma_1^2\,\tau^c + \sigma_2^2\,(1-\tau^c)\}}}\,\frac{\left(V_B^{-1}\,W_B\right)_{[3,1]}}{\sqrt{\left(V_B^{-1}\right)_{[3,3]}}}$$

where V_B is a (3×3) symmetric matrix, and W_B is a (3×1) matrix given by:

$$\begin{aligned}
V_B[1,1] &= \frac{1}{12}\\
V_B[1,2] &= \frac{1}{12}\left[1 - 12\,\tau^c + 9\,(\tau^c)^2 + 2\,(\tau^c)^3\right]\\
V_B[1,3] &= \sigma_1\left[B_3 - \frac{1}{2}\,B_1\right] + \sigma_2\left[B_4 - \frac{1}{2}\,B_2\right] + \frac{1}{2}\,(\sigma_1 - \sigma_2)\,\tau^c(1-\tau^c)\,W(\tau^c)\\
V_B[2,2] &= \frac{1}{12}\left[1 - 6\,(\tau^c)^2 + 8\,(\tau^c)^3 - 3\,(\tau^c)^4\right]\\
V_B[2,3] &= \sigma_2\,B_4 - \frac{1}{2}\left[1 - (\tau^c)^2\right](\sigma_1\,B_1 + \sigma_2\,B_2) - \tau^c\,\sigma_2\,B_2 + \frac{1}{2}\,\tau^c\,(1-\tau^c)^2\,(\sigma_1 - \sigma_2)\,W(\tau^c)\\
V_B[3,3] &= \sigma_1^2\,B_5 + \sigma_2^2\,B_6 - (\sigma_1\,B_1 + \sigma_2\,B_2)^2 + (\sigma_1 - \sigma_2)^2\,\tau^c\,(1-\tau^c)\,W(\tau^c)^2\\
&\quad +2\,(\sigma_1 - \sigma_2)\;W(\tau^c)\;\{\tau^c\,\sigma_2\,B_2 - (1-\tau^c)\,\sigma_1\,B_1\}
\end{aligned}$$

$$\begin{aligned}
W_B[1,1] &= \sigma_1\,\{\tau^c\,W(\tau^c) - B_1\} - \frac{1}{2}\,\sigma_1\,W(\tau^c) + \sigma_2\;\{W(1) - \tau^c\,W(\tau^c) - B_2\}\\
&\quad -\frac{1}{2}\,\sigma_2\;\{W(1) - W(\tau^c)\}\\
W_B[2,1] &= \sigma_2\;\{W(1) - \tau^c\,W(\tau^c) - B_2\} - \frac{1}{2}\,\sigma_1\,(1-\tau^c)^2\,W(\tau^c)\\
&\quad -\frac{1}{2}\,\sigma_2\left[1 + (\tau^c)^2\right]\{W(1) - W(\tau^c)\}\\
W_B[3,1] &= \frac{1}{2}\,\sigma_1^2\left\{W(\tau^c)^2 - \tau^c\right\} + \frac{1}{2}\,\sigma_2^2\left\{W(1)^2 - W(\tau^c)^2 - (1-\tau^c)\right\}\\
&\quad -\left[\sigma_1\,W(\tau^c) + \sigma_2\;\{W(1) - W(\tau^c)\}\right]\;\{\sigma_1\,B_1 + \sigma_2\,B_2\}\\
&\quad -\sigma_1\,(\sigma_1 - \sigma_2)\,(1-\tau^c)\;W(\tau^c)^2 + \sigma_2\,(\sigma_1 - \sigma_2)\,\tau^c\,W(\tau^c)\;\{W(1) - W(\tau^c)\}
\end{aligned}$$

Theorem 3: Suppose the true data generating process for the time series $\{y_t\}_{t=0}^{T}$ is given by equations (4) and (5). The limiting distribution of t-statistic for $H_o : \rho = 1$, denoted by t_P^C, based on the OLS regression (3) is given by:

$$t_P^C \Rightarrow \frac{1}{\sqrt{\{\sigma_1^2\,\tau^c + \sigma_2^2\,(1-\tau^c)\}}}\,\frac{\left(V_C^{-1}\,W_C\right)_{[3,1]}}{\sqrt{\left(V_C^{-1}\right)_{[3,3]}}}$$

where V_C is a (3×3) symmetric matrix, and W_C is a (3×1) matrix given by:

$$
\begin{aligned}
V_C[1,1] &= \frac{1}{12}\left[1 - 3\,\tau^c + 3\,(\tau^c)^2\right] \\
V_C[1,2] &= \frac{1}{12}\,(1-\tau^c)^3 \\
V_C[1,3] &= \sigma_1\,B_3 + \sigma_2\,B_4 - \frac{1}{2}\,\tau^c\,\sigma_1\,B_1 - \frac{1}{2}\,(1+\tau^c)\,\sigma_2\,B_2 \\
V_C[2,2] &= \frac{1}{12}\,(1-\tau^c)^3 \\
V_C[2,3] &= \sigma_2\,B_4 - \frac{1}{2}\,(1+\tau^c)\,\sigma_2\,B_2 \\
V_C[3,3] &= \sigma_1^2\,B_5 - (\tau^c)^{-1}\,\sigma_1^2\,B_1^2 + \sigma_2^2\,B_6 - (1-\tau^c)^{-1}\,\sigma_2^2\,B_2^2
\end{aligned}
$$

$$
\begin{aligned}
W_C[1,1] &= \frac{1}{2}\,\sigma_1\,\tau^c\,W(\tau^c) + \frac{1}{2}\,\sigma_2\,(1-\tau^c)\,\{W(1) - W(\tau^c)\} - \sigma_1\,B_1 - \sigma_2\,B_2 \\
W_C[2,1] & \quad \frac{1}{2}\,\sigma_2\,(1-\tau^c)\,\{W(1) - W(\tau^c)\} - \sigma_2\,B_2 \\
W_C[3,1] &= \frac{1}{2}\,\sigma_1^2\,\left\{W(\tau^c)^2 - \tau^c\right\} - (\tau^c)^{-1}\,\sigma_1^2\,W(\tau^c)\,B_1 \\
&\quad + \frac{1}{2}\,\sigma_2^2\,\left\{W(1)^2 - W(\tau^c)^2 - (1-\tau^c)\right\} - (1-\tau^c)^{-1}\,\sigma_2^2\,\{W(1) - W(\tau^c)\}\,B_2
\end{aligned}
$$

The limiting null distributions of t_P^i (i=A,B,C) depends on the break-fraction (τ^c), the pre-break variance (σ_1^2), and the post-break variance (σ_2^2). Therefore, we assess the size of t_P^i (i=A,B,C) using finite sample simulations based on the data generating process described by (4) and (5). We set $\gamma = 0$, and $\mu_0 = 0$. The error term is generated as follows: $\epsilon_t = \left\{1_{[t \le \tau^c T]} + \frac{\sigma_2}{\sigma_1}\,1_{[t > \tau^c T]}\right\}\eta_t$, and $\eta_t \sim i.i.d.N(0,1)$, and used all combinations arising from $\tau^c = \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9\}$ and $\frac{\sigma_2}{\sigma_1} = \{4, 2.5, 1.67, 1.25, 1, 0.8, 0.6, 0.4, 0.25\}$. We used 10,000 replications for two different sample sizes, namely, $T = \{100, 200\}$. The empirical size of all statistics is calculated based on the critical values tabulated in Perron (1989) at the 5% significance level. The empirical size of t_P^A are presented in Tables 1 and 2, of t_P^B in Tables 3 and 4, and of t_P^C in Tables 5 and 6. A general pattern emerges from these simulations which shows that the statistics are over-sized if either there is a fall in the innovation variance relatively early in the sample or if there is an increase in the innovation variance relatively late in the sample.

3. Modified Perron Unit Root Tests

Given that Perron's (1989) unit root statistics can be over-sized in the presence of an innovation break, we adapt the modified GLS strategy proposed by Kim, Leybourne, and Newbold (2002) to develop modified Perron tests that do not suffer such size distortions. First, we estimate the pre-break variance and the post-break variance based on the estimated residuals $\{\hat{\epsilon}_t^i\}$ (i=A,B,C) from regressions (1)-(3) respectively as follows:

$$\hat{\sigma}_1^{i\,2} = \frac{1}{\tau^c T} \sum_{t=1}^{\tau^c T} \hat{\epsilon}_t^{i\,2} \tag{6}$$

$$\hat{\sigma}_2^{i\,2} = \frac{1}{(T - \tau^c T)} \sum_{t=\tau^c+1}^{T} \hat{\epsilon}_t^{i\,2} \tag{7}$$

Next, we apply the following transformation to the dependent variable:

$$\tilde{y}_t^i = \frac{y_t}{\hat{\sigma}_1^i} 1_{(1 \leq \tau^c T)} + \frac{y_t}{\hat{\sigma}_2^i} 1_{(1 \geq \tau^c T+1)} \tag{8}$$

The GLS transformation based on (8) when applied to regressions (1)-(3) yields:

$$\tilde{y}_t^A = \begin{cases} \frac{\mu_0}{\hat{\sigma}_1^A} + \frac{\mu_2}{\hat{\sigma}_1^A} t + \rho \tilde{y}_{t-1}^A + \frac{\epsilon_t}{\hat{\sigma}_1^A} & \text{if } t \leq \tau^c T \\ \frac{(\mu_0 + \mu_1 + \gamma)}{\hat{\sigma}_2^A} + \frac{\mu_2}{\hat{\sigma}_2^A} t + \rho \tilde{y}_{t-1}^A + \left(\frac{\hat{\sigma}_1^A - \hat{\sigma}_2^A}{\hat{\sigma}_2^A} \right) \tilde{y}_{t-1}^A + \frac{\epsilon_t}{\hat{\sigma}_1^A} & \text{if } t = \tau^c T + 1 \\ \frac{(\mu_0 + \mu_1)}{\hat{\sigma}_2^A} + \frac{\mu_2}{\hat{\sigma}_2^A} t + \rho \tilde{y}_{t-1}^A + \frac{\epsilon_t}{\hat{\sigma}_2^A} & \text{if } t \geq \tau^c T + 2 \end{cases}$$

$$\tilde{y}_t^B = \begin{cases} \frac{\mu_0}{\hat{\sigma}_1^B} + \frac{\mu_2}{\hat{\sigma}_1^B} t + \rho \tilde{y}_{t-1}^B + \frac{\epsilon_t}{\hat{\sigma}_1^B} & \text{if } t \leq \tau^c T \\ \frac{\mu_0}{\hat{\sigma}_2^B} + \frac{\mu_2}{\hat{\sigma}_2^B} t + \frac{\mu_3}{\hat{\sigma}_2^B} + \rho \tilde{y}_{t-1}^B + \left(\frac{\hat{\sigma}_1^B - \hat{\sigma}_2^B}{\hat{\sigma}_2^B} \right) \tilde{y}_{t-1}^B + \frac{\epsilon_t}{\hat{\sigma}_1^B} & \text{if } t = \tau^c T + 1 \\ \frac{\mu_0}{\hat{\sigma}_2^B} + \frac{\mu_2}{\hat{\sigma}_2^B} t + \frac{\mu_3}{\hat{\sigma}_2^B} (t - \tau^c T) + \rho \tilde{y}_{t-1}^B + \frac{\epsilon_t}{\hat{\sigma}_2^B} & \text{if } t \geq \tau^c T + 2 \end{cases}$$

$$\tilde{y}_t^C = \begin{cases} \frac{\mu_0}{\hat{\sigma}_1^C} + \frac{\mu_2}{\hat{\sigma}_1^C} t + \rho \tilde{y}_{t-1}^C + \frac{\epsilon_t}{\hat{\sigma}_1^C} & \text{if } t \leq \tau^c T \\ \frac{(\mu_0 + \mu_1 + \gamma)}{\hat{\sigma}_2^C} + \frac{(\mu_2 + \mu_3)}{\hat{\sigma}_2^c} t + \rho \tilde{y}_{t-1}^C + \left(\frac{\hat{\sigma}_1^C - \hat{\sigma}_2^C}{\hat{\sigma}_2^C} \right) \tilde{y}_{t-1}^C + \frac{\epsilon_t}{\hat{\sigma}_1^C} & \text{if } t = \tau^c T + 1 \\ \frac{(\mu_0 + \mu_1)}{\hat{\sigma}_2^C} + \frac{(\mu_2 + \mu_3)}{\hat{\sigma}_2^C} t + \rho \tilde{y}_{t-1}^C + \frac{\epsilon_t}{\hat{\sigma}_2^C} & \text{if } t \geq \tau^c T + 2 \end{cases}$$

The modified unit root statistics, denoted by t_P^{i*} (i=A,B,C), are defined as the t-statistic for the null hypothesis $H_o : \rho = 1$ in the following OLS regressions:

$$\tilde{y}_t^i = \alpha_0 + \alpha_1 DU_t^c + \beta D_t^c + \alpha_2 t + \alpha_3 DT_t^c + \rho \tilde{y}_{t-1}^i + \eta_t \tag{9}$$

(i=A,B,C) with $\tilde{y}_t^i$ defined in (8). The dummy variable D_t^c removes the observation corresponding to $t = \tau^c T + 1$ as it does not belong to either the pre-break or post-break regimes. If k^* additional lagged first differences ($\{\Delta y_{t-j}\}_{j=1}^{k^*}$) are included in regression (1)-(3), then additional dummy variables need to be included in regression (9) to remove the middle k^*+1 observations corresponding to $t = \{\tau^c T+1, \tau^c T+2, \ldots, \tau^c T+(k^*+1)\}$. Regression (9) is the same as the Mixed model regression of Perron (1989), and so the critical values of t_P^{i*} (i=A,B,C) are the same as those of t_P^C given in Table VI.B of Perron (1989).

We assessed the empirical size of the modified Perron (1989) tests, t_P^{i*} (i=A,B,C) using the simulation design described in Section 2. The results for t_P^{A*} are shown in Tables 7 and 8, for t_P^{B*} are shown in Tables 9 and 10, and for t_P^{C*} are shown in Tables 11 and

12. Based on the corresponding 5% level critical values tabulated in Perron (1989), the empirical size of t_P^{i*} (i=A,B,C) are close to the nominal size in all cases. Therefore, the modified Perron tests correct the size distortions present in the Perron tests.

4. Application to Nelson-Plosser Macroeconomic Series

Perron (1989) used his statistics, t_P^i (i=A,C), to test for the presence of a unit root in the thirteen macroeconomic series contained in the Nelson and Plosser (1982) data set. Perron (1989) specified the Crash model for all series except the Common Stock Price series and the Real Wages series. For the latter two series, Perron (1989) used the Mixed model. The empirical evidence, based on Perron's (1989) testing strategy, reveals substantial evidence in favour of the trend-break stationary alternative. Specifically, Perron (1989) rejected the unit root null hypothesis for all series except the Consumer Price Index, the Velocity, and the Interest Rate series.[4] [4] However, Perron's (1989) analysis did not take into account the potential presence of a break in the innovation variance. The asymptotic distribution of Perron's (1989) statistics, together with their finite sample simulation performance, show that these statistics can be over-sized when there is a break in the innovation variance under the unit root null hypothesis. Therefore, we re-evaluate the empirical evidence by allowing for the presence of a break in the innovation variance.

We begin by estimating the pre-break standard deviation of the innovation term ($\hat{\sigma}_1^i$, i=A,C), and the post-break standard deviation of the innovation term ($\hat{\sigma}_2^i$, i=A,C) based on regressions (1) and (3) for the Crash model and Mixed model specifications respectively. The results, shown in Table 13, indicate that there may be substantial changes in the innovation variance across sub-samples as measured by the ratio of the estimated post-break innovation standard deviation to the estimated pre-break innovation standard deviation ($\hat{\sigma}_2^i/\hat{\sigma}_1^i$). While the innovation variance for the Employment series and the Interest Rate series are fairly constant across sub-samples, there is an increase in the innovation variance for the Common Stock Price series and the Velocity series. For all other series, there is a decrease in the innovation variance. Therefore, our preliminary evidence suggests that one should incorporate a potential break in the innovation variance when testing for the presence of a unit root.

We calculate the modified Perron statistic, t_P^{i*} (i=A,C) based on regression (9), see Table 13 below. Since the asymptotic null distribution of the modified Perron statistics are the same as that of Perron's (1989) Mixed model statistic (t_P^C), we extrapolated the critical values for t_P^{i*} (i=A,C) based on the tabulated critical values in Table VI.B of Perron (1989). We reject the unit root null hypothesis for only four series: Real GNP at the 1% level, Nominal GNP at the 5% level, Industrial Production at the 2.5% level, and Employment at the 10% level. While Perron (1989) rejected the unit root null hypothesis for ten out of the thirteen series, our evidence against the unit root null hypothesis, in comparison, is

[4] The reader is referred to Table VII of Perron (1989), pp. 1383 for further details.

much weaker. It follows that substantially strong empirical evidence against the unit root null found by Perron (1989) is a result of his statistic being over-sized owing to a break in the innovation variance.

5. Conclusion

Perron (1989) developed unit root tests that are specifically designed to have power against the stationary alternative characterized by a break in the trend function. Following Perron (1989), we consider three different characterizations of the break under the stationary alternative: the Crash model that allows for a break in the mean; the Changing Growth model that allows for a break in the time trend with the two segments joined at the time of the break; and the Mixed model that allows for a simultaneous break in the mean and the time trend parameters. We assume that the break in the innovation variance, if it occurs, coincides with the known break-date for the trend function parameters.

Our assessment of the size of Perron's (1989) unit root tests show that they can suffer from serious size distortions. We show that the limiting null distribution of Perron's statistics depend on the pre-break variance, the post-break variance, and the break-fraction. Our simulation evidence reveals that, in the presence of a break in the innovation variance, the Perron tests are over-sized when there is a decrease in the innovation variance relatively early in the sample or when there is an increase in the innovation variance relatively late in the sample. Therefore, we propose modified Perron tests that are based on the GLS transformation proposed by Kim, Leybourne, and Newbold (2002). The modified Perron tests have the same limiting null distribution as Perron's Mixed model statistic, and so the practitioner can use the critical values tabulated in Perron (1989) for empirical applications. Simulation evidence confirms that the modified Perron tests maintain their size in the presence of a break in the innovation variance.

We illustrate the use of the modified Perron test with an application to the Nelson and Plosser (1982) macroeconomic series. Our empirical evidence shows that most series can be characterized by a break in the innovation variance, and incorporating this innovation break substantially weakens the evidence against the unit root null hypothesis compared to the evidence presented in Perron (1989). Therefore, the practitioner should evaluate the innovation variance before testing for the presence of a unit root. In the eventuality that a break in the innovation variance is suspected, the practitioner should use the modified Perron test given that it maintains its size and continues to have power against the trend-break stationary alternative.

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Appendix

In what follows, we outline the proofs of Theorems 1-3. The location of the break-date is $T_b^c = [\tau^c T]$. All summations are taken over the sample, that is, from t=1 to T unless otherwise specified. The results are based on the functional weak convergence result $T^{-1/2} \sum_{t=1}^{[r\,T]} \eta_t \Rightarrow W(r) \;\forall\, r \in [0,1]$ where is $W(r)$ is the Wiener Process defined on the unit interval, and "$\Rightarrow$" denotes weak convergence. Based on the data generating process given in (1), and assuming without loss of generality that $\mu_0 = 0$, $\gamma = 0$ and $y_0 = 0$, we can show that:

$$\text{(A.1)} \qquad T^{-1/2} \sum_{t=1}^{T_b^c} \epsilon_t \Rightarrow \sigma_1 \, W(\tau^c)$$

$$\text{(A.2)} \qquad T^{-1/2} \sum_{t=T_b^c+2}^{T} \epsilon_t \Rightarrow \sigma_2 \, \{W(1) - W(\tau^c)\}$$

$$\text{(A.3)} \qquad T^{-1} \sum_{t=1}^{T_b^c} y_{t-1}\, \epsilon_t \Rightarrow \frac{1}{2}\, \sigma_1^2 \, \Big\{ W(\tau^c)^2 - \tau^c \Big\}$$

$$\text{(A.4)} \qquad T^{-1} \sum_{t=T_b^c+2}^{T} y_{t-1}\, \epsilon_t \Rightarrow \; \frac{1}{2}\, \sigma_2^2 \, \Big\{ W(1)^2 - 1 \Big\} - \frac{1}{2}\, \sigma_2^2 \, \Big\{ W(\tau^c)^2 - \tau^c \Big\} + \; \sigma_2 \, (\sigma_1 - \sigma_2) \, W(\tau^c) \, \{W(1) - W(\tau^c)\}$$

$$(A.5) \qquad T^{-3/2} \sum_{t=1}^{T_b^c} y_{t-1} \Rightarrow \sigma_1 \int_0^{\tau^c} W(r)\, dr$$

$$(A.6) \qquad T^{-3/2} \sum_{t=T_b^c+2}^{T} y_{t-1} \Rightarrow \sigma_2 \int_{\tau^c}^{1} W(r)\, dr + (\sigma_1 - \sigma_2)\,(1-\tau^c)\, W(\tau^c)$$

$$\text{(A.7)} \qquad T^{-2} \sum_{t=1}^{T_b^c} y_{t-1}^2 \Rightarrow \sigma_1^2 \int_0^{\tau^c} W(r)^2\, dr$$

$$\text{(A.8)} \qquad T^{-2} \sum_{t=T_b^c+2}^{T} y_{t-1}^2 \Rightarrow \sigma_2^2 \int_{\tau^c}^{1} W(r)^2\, dr + (\sigma_1 - \sigma_2)^2\,(1-\tau^c)\, W(\tau^c)^2 + 2\,(\sigma_1-\sigma_2)\,\sigma_2\, W(\tau^c) \int_{\tau^c}^{1} W(r)\, dr$$

$$\text{(A.9)} \qquad T^{-3/2} \sum_{t=1}^{T_b^c} t\, e_t \Rightarrow \sigma_1 \left[\tau^c\, W(\tau^c) - \int_0^{\tau^c} W(r)\, dr\right]$$

$$\text{(A.10)} \qquad T^{-3/2} \sum_{t=T_b^c+2}^{T} t\, e_t \Rightarrow \sigma_2 \left[W(1) - \tau^c\, W(\tau^c) - \int_{\tau^c}^{1} W(r)\, dr\right]$$

$$\text{(A.11)} \qquad T^{-5/2} \sum_{t=1}^{T_b^c} t\, y_{t-1} \Rightarrow \sigma_1 \int_0^{\tau^c} r\, W(r)\, dr$$

$$\text{(A.12)} \qquad T^{-5/2} \sum_{t=T_b^c+2}^{T} t\, y_{t-1} \Rightarrow \sigma_2 \int_{\tau^c}^{1} r\, W(r)\, dr + \frac{1}{2}\,(\sigma_1-\sigma_2)\, \left(1-(\tau^c)^2\right)\, W(\tau^c)$$

Regressions (1)-(3) can be written as:

$$\text{(A.13)} \qquad Y = X_i\, \theta_i + \epsilon$$

with i=A,B,C corresponding to regressions (1), (2), and (3) respectively, and $Y = [y_t]$, $\epsilon = [\epsilon_t]$, $X_A = [1, DU_t, D_t, t, y_{t-1}]$, $\theta_A = (\mu_0, \mu_1, \gamma, \mu_2, \rho_A)'$, $X_B = [1, t, DT_t^*, y_{t-1}]$, $\theta_B = (\mu_0, \mu_2, \mu_3, \rho_B)'$, $X_C = [1, DU_t, D_t, t, DT_t, y_{t-1}]$, and $\theta_C = (\mu_0, \mu_1, \gamma, \mu_2, \mu_3, \rho_C)'$. We partition the matrix X_i as $[X_{i,1}|X_{i,2}]$ for i=A,B,C with $X_{A,1} = [1, DU_t, D_t]$, $X_{A,2} = [t, y_{t-1}]$, $X_{B,1} = [1]$, $X_{B,2} = [t, DT_t^*, y_{t-1}]$, $X_{C,1} = [1, DU_t, D_t]$, and $X_{C,2} = [t, DT_t, y_{t-1}]$. It follows that regression (A.13) will yield numerically equivalent results as:

$$\text{(A.14)} \qquad Y_i^* = X_{i,2}^*\, \theta_{i,2} + \epsilon_i^*$$

for i=A,B,C where Y_i^*, $X_{i,2}^*$, ϵ_i^* are respectively the projections of Y, $X_{i,2}$, and ϵ on the space spanned by the columns of $X_{i,1}$, $\theta_{A,2} = (\mu_2, \rho_A)'$, $\theta_{B,2} = (\mu_2, \mu_3, \rho_B)'$, $\theta_{C,2} =$

$(\mu_2, \mu_3, \rho_C)'$. The OLS estimator of $\theta_{i,2}$ is equal to $\hat{\theta}_{i,2} = \left(X_{i,2}^{*'} X_{i,2}^{*}\right)^{-1} X_{i,2}^{*'} Y_i^*$ which can also be written as:

$$D_{i,T}\,(\hat{\theta}_{i,2} - \theta_{i,2}) = \left[D_{i,T}^{-1}\,(X_{i,2}^{*'} X_{i,2}^{*})\,D_{i,T}^{-1}\right]^{-1}\left[D_{i,T}^{-1}\,X_{i,2}^{*'}\epsilon_i^*\right] \tag{A.15}$$

where $D_{A,T} = diag(T^{3/2}, T)$, $D_{B,T} = diag(T^{3/2}, T^{3/2}, T)$, and $D_{C,T} = diag(T^{3/2}, T^{3/2}, T)$.

Also, the estimated error variance from regression (1)-(3) are given by:

$$\hat{\sigma}_i^2 = \frac{1}{(T - n_i)}\left\{\epsilon_i^{*'}\epsilon_i^* - \left[D_{i,T}^{-1}\,X_{i,2}^{*'}\epsilon_i^*\right]'\left[D_{i,T}^{-1}\,(X_{i,2}^{*'} X_{i,2}^{*})\,D_{i,T}^{-1}\right]^{-1}\left[D_{i,T}^{-1}\,X_{i,2}^{*'}\epsilon_i^*\right]\right\} \tag{A.16}$$

for i=A,B,C, $n_1 = 2$, and $n_2 = n_3 = 3$.

Define the $(n_i \times n_i)$ symmetric matrix $V_{i,T} = D_{i,T}^{-1}\,(X_{i,2}^{*'} X_{i,2}^{*})\,D_{i,T}^{-1}$, and the $(n_i \times 1)$ matrix $W_{i,T} = D_{i,T}^{-1}\,X_{i,2}^{*'}\epsilon_i^*$ for i=A,B,C. We can show that:

$$\begin{aligned}
V_{A,T}[1,1] &= T^{-3}\sum_{t=1}^{\tau^c T}(t-\bar{t}_1)^2 + T^{-3}\sum_{t=\tau^c T+2}^{T}(t-\bar{t}_2)^2 \\
V_{A,T}[1,2] &= \left(T^{-5/2}\sum_{t=1}^{\tau^c T} t\,y_{t-1}\right) - \left(T^{-1}\,\bar{t}_1\right)\left(T^{-3/2}\sum_{t=1}^{\tau^c T} y_{t-1}\right) \\
&\quad + \left(T^{-5/2}\sum_{t=\tau^c T+2}^{T} t\,y_{t-1}\right) - \left(T^{-1}\,\bar{t}_2\right)\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} y_{t-1}\right) \\
V_{A,T}[2,2] &= \left(T^{-2}\sum_{t=1}^{\tau^c T} y_{t-1}^2\right) - T^{-2}\,(\tau^c T)^{-1}\left(\sum_{t=1}^{\tau^c T} y_{t-1}\right)^2 \\
&\quad + \left(T^{-2}\sum_{t=\tau^c T+2}^{T} y_{t-1}^2\right) - T^{-2}\,(T - \tau^c T - 1)^{-1}\left(\sum_{t=\tau^c T+2}^{T} y_{t-1}\right)^2
\end{aligned}$$

$$\begin{aligned}
W_{A,T}[1,1] &= \left(T^{-3/2}\sum_{t=1}^{\tau^c T} t\,\epsilon_t\right) - \left(T^{-1}\,\bar{t}_1\right)\left(T^{-1/2}\sum_{t=1}^{\tau^c T}\epsilon_t\right) \\
&\quad + \left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} t\,\epsilon_t\right) - \left(T^{-1}\,\bar{t}_2\right)\left(T^{-1/2}\sum_{t=\tau^c T+2}^{T}\epsilon_t\right) \\
W_{A,T}[2,1] &= \left(T^{-1}\sum_{t=1}^{\tau^c T} y_{t-1}\,\epsilon_t\right) - (\tau^c)^{-1}\left(T^{-3/2}\sum_{t=1}^{\tau^c T} y_{t-1}\right)\left(T^{-1/2}\sum_{t=1}^{\tau^c T}\epsilon_t\right) \\
&\quad + \left(T^{-1}\sum_{t=\tau^c T+2}^{T} y_{t-1}\,\epsilon_t\right) - (1-\tau^c)^{-1}\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} y_{t-1}\right)\left(T^{-1/2}\sum_{t=\tau^c T+2}^{T}\epsilon_t\right)
\end{aligned}$$

$$
\begin{aligned}
V_{B,T}[1,1] &= T^{-3}\sum_{t=1}^{T}(t-\bar{t})^2 \\
V_{B,T}[1,2] &= T^{-3}\sum_{t=1}^{\tau^c T}(t-\bar{t})\,(0-\bar{t}_3) + T^{-3}\sum_{t=\tau^c T+1}^{T}(t-\bar{t})\,(t-\tau^c T-\bar{t}_3) \\
V_{B,T}[1,3] &= \left(T^{-5/2}\sum_{t=1}^{T} t y_{t-1}\right) - \frac{1}{2}\left(1+\frac{1}{T}\right)\left(T^{-3/2}\sum_{t=1}^{T} y_{t-1}\right) \\
V_{B,T}[2,2] &= T^{-3}\sum_{t=1}^{\tau^c T}(0-\bar{t}_3)^2 + T^{-3}\sum_{t=\tau^c T+1}^{T}(t-\tau^c T-\bar{t}_3)^2 \\
V_{B,T}[2,3] &= \left(T^{-5/2}\sum_{t=\tau^c T+1}^{T} t y_{t-1}\right) - T^{-2}\left(\frac{T(T+1)}{2} - \frac{\tau^c T(\tau^c T+1)}{2}\right)\left(T^{-3/2}\sum_{t=1}^{T} y_{t-1}\right) \\
&\quad -\tau^c\left(T^{-3/2}\sum_{t=\tau^c T+1}^{T} y_{t-1}\right) + \tau^c(1-\tau^c)\left(T^{-3/2}\sum_{t=1}^{T} y_{t-1}\right) \\
V_{B,T}[3,3] &= \left(T^{-2}\sum_{t=1}^{T} y_{t-1}^2\right) - \left(T^{-3/2}\sum_{t=1}^{T} y_{t-1}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
W_{B,T}[1,1] &= \left(T^{-5/2}\sum_{t=1}^{T} t\epsilon_t\right) - \frac{1}{2}\left(1+\frac{1}{T}\right)\left(T^{-3/2}\sum_{t=1}^{T}\epsilon_t\right) \\
W_{B,T}[2,1] &= T^{-3/2}\sum_{t=\tau^c T+1}^{T} t\epsilon_t + \left(T^{-1/2}\sum_{t=1}^{\tau^c T}\epsilon_t\right)\left[-\frac{1}{2}(1-\tau^c)^2 T - \frac{1}{2}(1-\tau^c)\right] \\
&\quad + \left(T^{-1/2}\sum_{t=\tau^c T+1}^{T}\epsilon_t\right)\left[\left\{-\tau^c - \frac{1}{2}(1-\tau^c)^2\right\} T - \frac{1}{2}(1-\tau^c)\right] \\
W_{B,T}[3,1] &= \left(T^{-1}\sum_{t=1}^{T} y_{t-1}\epsilon_t\right) - \left(T^{-1/2}\sum_{t=1}^{T}\epsilon_t\right)\left(T^{-3/2}\sum_{t=1}^{T} y_{t-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
V_{C,T}[1,1] &= T^{-3}\sum_{t=1}^{\tau^c T}(t-\bar{t}_1)^2 + T^{-3}\sum_{t=\tau^c T+2}^{T}(t-\bar{t}_2)^2 \\
V_{C,T}[1,2] &= T^{-3}\sum_{t=\tau^c T+2}^{T}(t-\bar{t}_2)^2
\end{aligned}
$$

$$
V_{C,T}[1,3] = \left(T^{-5/2}\sum_{t=1}^{\tau^c T} t y_{t-1}\right) - \left(T^{-1}\,\bar{t}_1\right)\left(T^{-3/2}\sum_{t=1}^{\tau^c T} y_{t-1}\right)
$$

$$\left(T^{-5/2}\sum_{t=\tau^c T+2}^{T} t y_{t-1}\right) - \left(T^{-1}\,\bar{t}_2\right)\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} y_{t-1}\right)$$

$$V_{C,T}[2,2] = T^{-3}\sum_{t=\tau^c T+2}^{T}(t-\bar{t}_2)^2$$

$$V_{C,T}[2,3] = \left(T^{-5/2}\sum_{t=\tau^c T+2}^{T} t y_{t-1}\right) - \left(T^{-1}\,\bar{t}_2\right)\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} y_{t-1}\right)$$

$$V_{C,T}[3,3] = \left(T^{-2}\sum_{t=1}^{\tau^c T} y_{t-1}^2\right) - (\tau^c)^{-1}\left(T^{-3/2}\sum_{t=1}^{\tau^c T} y_{t-1}\right)^2$$

$$\left(T^{-2}\sum_{t=\tau^c T+2}^{T} y_{t-1}^2\right) - (1-\tau^c)^{-1}\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} y_{t-1}\right)^2$$

and

$$W_{C,T}[1,1] = \left(T^{-3/2}\sum_{t=1}^{\tau^c T} t\epsilon_t\right) - \left(T^{-1}\bar{t}_1\right)\left(T^{-1/2}\sum_{t=1}^{\tau^c T}\epsilon_t\right)$$

$$\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} t\epsilon_t\right) - \left(T^{-1}\bar{t}_2\right)\left(T^{-1/2}\sum_{t=\tau^c T+2}^{T}\epsilon_t\right)$$

$$W_{C,T}[2,1] = \left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} t\epsilon_t\right) - \left(T^{-1}\bar{t}_2\right)\left(T^{-1/2}\sum_{t=\tau^c T+2}^{T}\epsilon_t\right)$$

$$W_{C,T}[3,1] = \left(T^{-1}\sum_{t=1}^{\tau^c T} y_{t-1}\epsilon_t\right) - (\tau^c)^{-1}\left(T^{-3/2}\sum_{t=1}^{\tau^c T} y_{t-1}\right)\left(T^{-1/2}\sum_{t=1}^{\tau^c T}\epsilon_t\right)$$

$$\left(T^{-1}\sum_{t=\tau^c T+2}^{T} y_{t-1}\epsilon_t\right) - (1-\tau^c)^{-1}\left(T^{-3/2}\sum_{t=\tau^c T+2}^{T} y_{t-1}\right)\left(T^{-1/2}\sum_{t=\tau^c T+2}^{T}\epsilon_t\right)$$

where $\bar{t}_1 = (\tau^c T)^{-1}\sum_{t=1}^{\tau^c T} t$, $\bar{t}_2 = (T-\tau^c T-1)^{-1}\sum_{t=\tau^c T+2}^{T} t$, $\bar{t} = T^{-1}\sum_{t=1}^{T} t$, and $\bar{t}_3 = T^{-1}\sum_{t=1}^{T-\tau^c T} t$. Based on the limiting behaviour of the moments in (A.1)-(A.12), tedious calculations imply that $V_{i,T} \Rightarrow V_i$, and $W_{i,T} \Rightarrow W_i$ for i=A,B,C (as described in Theorems 1-3), and so expression (A.15) implies that:

(A.17) $$D_{i,T}\,(\hat{\theta}_{i,2} - \theta_{i,2}) \Rightarrow V_i^{-1}\,W_i$$

Also, equation (A.16) implies that $\hat{\sigma}_i^2$ is asymptotically equivalent to $\frac{1}{T}\,\epsilon_i^{*'}\epsilon_i^* = T^{-1}\sum_{t=1}^{\tau^c T}(\epsilon_t - \bar{\epsilon}_1)^2 + T^{-1}\sum_{t=\tau^c T+2}^{T}(\epsilon_t - \bar{\epsilon}_2)^2$ where $\bar{\epsilon}_1 = (\tau^c T)^{-1}\sum_{t=1}^{\tau^c T}\epsilon_t$ and $\bar{\epsilon}_2 = (T-\tau^c T-1)^{-1}\sum_{t=\tau^c T+2}^{T}\epsilon_t$. Therefore,

$$\hat{\sigma}_i^2 \Rightarrow \tau^c\,\sigma_1^2 + (1-\tau^c)\,\sigma_2^2 \tag{A.18}$$

for i=A,B,C. Finally, the limiting distribution of $v\hat{a}r(\hat{\theta}_{i,2}) = \hat{\sigma}_i^2\left(X_{i,2}^{*\prime}X_{i,2}^{*}\right)^{-1}$ can be expressed as:

$$D_{i,T}\left[v\hat{a}r(\hat{\theta}_{i,2})\right]D_{i,T} \Rightarrow \left\{\tau^c\,\sigma_1^2 + (1-\tau^c)\,\sigma_2^2\right\}\,V_i^{-1} \tag{A.19}$$

Therefore, the limiting null distribution of the t-statistic t_P^i (i=A,B,C) based on regressions (1)-(3) respectively, namely:

$$t_P^i = \frac{T\,(\hat{\rho}_i - 1)}{\sqrt{\hat{\sigma}_i^2}\,\sqrt{T^2\left(X_{i,2}^{*\prime}X_{i,2}^{*}\right)^{-1}_{[n_i,n_i]}}}$$

can be obtained by combining expressions (A.17), (A.18), and (A.19).

Table 1. Size of t_P^A with sample size $T = 100$, $\mu_0 = \gamma = 0$

τ^c	$\delta = \sigma_2/\sigma_1$								
	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.043	0.041	0.043	0.045	0.049	0.055	0.071	0.133	0.275
0.20	0.044	0.043	0.043	0.045	0.048	0.056	0.080	0.143	0.250
0.30	0.052	0.050	0.046	0.047	0.051	0.060	0.078	0.121	0.182
0.40	0.061	0.056	0.050	0.049	0.050	0.054	0.065	0.091	0.122
0.50	0.087	0.075	0.061	0.056	0.051	0.056	0.062	0.071	0.087
0.60	0.127	0.096	0.070	0.056	0.050	0.048	0.052	0.057	0.061
0.70	0.186	0.125	0.077	0.059	0.050	0.048	0.046	0.048	0.052
0.80	0.266	0.148	0.081	0.057	0.050	0.047	0.046	0.044	0.047
0.90	0.270	0.126	0.070	0.055	0.047	0.043	0.043	0.042	0.041

Table 2. Size of t_P^A with sample size $T = 200$, $\mu_0 = \gamma = 0$

τ^c	$\delta = \sigma_2/\sigma_1$								
	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.044	0.044	0.045	0.047	0.051	0.058	0.076	0.140	0.291
0.20	0.044	0.043	0.044	0.044	0.050	0.059	0.079	0.144	0.257
0.30	0.052	0.049	0.046	0.048	0.051	0.057	0.080	0.124	0.183
0.40	0.064	0.059	0.052	0.049	0.050	0.056	0.067	0.094	0.124
0.50	0.087	0.073	0.060	0.053	0.051	0.052	0.061	0.073	0.084
0.60	0.127	0.096	0.067	0.055	0.052	0.048	0.048	0.056	0.061
0.70	0.195	0.129	0.080	0.062	0.052	0.048	0.049	0.051	0.054
0.80	0.266	0.150	0.085	0.062	0.051	0.049	0.044	0.044	0.047
0.90	0.284	0.135	0.075	0.056	0.049	0.046	0.044	0.044	0.042

Table 3. Size of t_P^B with sample size $T = 100$, $\mu_0 = \gamma = 0$

τ^c	$\delta = \sigma_2/\sigma_1$								
	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.026	0.030	0.033	0.040	0.050	0.066	0.102	0.217	0.448
0.20	0.025	0.026	0.031	0.038	0.050	0.069	0.116	0.246	0.470
0.30	0.032	0.033	0.036	0.041	0.050	0.067	0.112	0.217	0.367
0.40	0.040	0.039	0.036	0.038	0.046	0.061	0.087	0.152	0.230
0.50	0.068	0.056	0.045	0.047	0.051	0.058	0.079	0.113	0.153
0.60	0.108	0.078	0.055	0.050	0.053	0.058	0.068	0.088	0.103
0.70	0.166	0.100	0.064	0.053	0.051	0.054	0.056	0.063	0.065
0.80	0.222	0.112	0.067	0.054	0.049	0.047	0.047	0.048	0.049
0.90	0.219	0.107	0.066	0.054	0.050	0.049	0.046	0.045	0.046

Table 4. Size of t_P^B with sample size $T = 200$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.028	0.030	0.034	0.039	0.049	0.066	0.100	0.211	0.454
0.20	0.025	0.028	0.030	0.037	0.046	0.069	0.116	0.242	0.476
0.30	0.032	0.033	0.036	0.042	0.052	0.070	0.115	0.218	0.369
0.40	0.042	0.039	0.039	0.041	0.047	0.062	0.092	0.159	0.237
0.50	0.066	0.055	0.046	0.044	0.048	0.057	0.078	0.116	0.152
0.60	0.109	0.079	0.057	0.052	0.052	0.059	0.069	0.088	0.104
0.70	0.167	0.102	0.065	0.053	0.052	0.053	0.057	0.065	0.069
0.80	0.230	0.115	0.068	0.054	0.051	0.048	0.049	0.049	0.050
0.90	0.227	0.104	0.065	0.054	0.051	0.046	0.046	0.043	0.045

Table 5. Size of t_P^C with sample size $T = 100$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.041	0.041	0.042	0.047	0.050	0.057	0.077	0.138	0.310
0.20	0.040	0.041	0.042	0.045	0.050	0.062	0.093	0.181	0.374
0.30	0.049	0.047	0.044	0.046	0.049	0.059	0.084	0.157	0.277
0.40	0.072	0.063	0.051	0.047	0.048	0.055	0.074	0.117	0.181
0.50	0.116	0.084	0.061	0.050	0.047	0.049	0.058	0.081	0.106
0.60	0.192	0.129	0.076	0.057	0.050	0.047	0.053	0.062	0.071
0.70	0.293	0.162	0.085	0.060	0.049	0.046	0.046	0.046	0.048
0.80	0.369	0.180	0.090	0.061	0.050	0.044	0.040	0.041	0.038
0.90	0.288	0.130	0.073	0.057	0.052	0.047	0.045	0.042	0.041

Table 6. Size of t_P^C with sample size $T = 200$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.041	0.041	0.042	0.046	0.052	0.058	0.079	0.146	0.334
0.20	0.039	0.041	0.040	0.044	0.050	0.060	0.091	0.182	0.379
0.30	0.053	0.047	0.045	0.047	0.050	0.061	0.086	0.160	0.287
0.40	0.074	0.066	0.053	0.050	0.049	0.057	0.076	0.126	0.192
0.50	0.114	0.085	0.064	0.051	0.049	0.050	0.064	0.087	0.115
0.60	0.189	0.124	0.075	0.056	0.050	0.048	0.051	0.061	0.070
0.70	0.294	0.164	0.087	0.058	0.050	0.046	0.046	0.046	0.047
0.80	0.377	0.179	0.087	0.057	0.048	0.044	0.039	0.038	0.036
0.90	0.330	0.143	0.076	0.056	0.050	0.044	0.042	0.039	0.038

Table 7. Size of t_P^{A*} with sample size $T = 100$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.051	0.049	0.050	0.050	0.050	0.051	0.050	0.049	0.051
0.20	0.046	0.048	0.049	0.049	0.048	0.050	0.050	0.052	0.053
0.30	0.047	0.049	0.048	0.048	0.049	0.051	0.051	0.050	0.052
0.40	0.050	0.048	0.048	0.048	0.046	0.046	0.047	0.048	0.049
0.50	0.054	0.051	0.051	0.051	0.050	0.051	0.053	0.050	0.053
0.60	0.053	0.049	0.050	0.050	0.050	0.050	0.051	0.050	0.051
0.70	0.052	0.051	0.050	0.050	0.050	0.050	0.048	0.048	0.049
0.80	0.053	0.051	0.049	0.049	0.049	0.049	0.050	0.048	0.048
0.90	0.057	0.051	0.052	0.054	0.053	0.050	0.054	0.052	0.052

Table 8. Size of t_P^{A*} with sample size $T = 200$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.050	0.051	0.051	0.052	0.052	0.051	0.051	0.050	0.053
0.20	0.048	0.049	0.051	0.050	0.050	0.050	0.050	0.050	0.053
0.30	0.053	0.051	0.052	0.053	0.054	0.052	0.054	0.053	0.053
0.40	0.050	0.049	0.049	0.049	0.048	0.050	0.049	0.048	0.049
0.50	0.050	0.048	0.048	0.048	0.048	0.047	0.048	0.050	0.051
0.60	0.054	0.053	0.052	0.053	0.053	0.051	0.052	0.054	0.053
0.70	0.054	0.051	0.052	0.052	0.053	0.051	0.050	0.051	0.050
0.80	0.053	0.052	0.053	0.053	0.052	0.054	0.050	0.050	0.049
0.90	0.052	0.048	0.050	0.052	0.050	0.050	0.051	0.051	0.049

Table 9. Size of t_P^{B*} with sample size $T = 100$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
	0.046	0.049	0.050	0.052	0.052	0.053	0.051	0.055	0.061
	0.048	0.050	0.051	0.052	0.053	0.053	0.055	0.057	0.059
	0.049	0.051	0.051	0.050	0.050	0.050	0.052	0.055	0.057
	0.051	0.050	0.051	0.050	0.051	0.050	0.049	0.051	0.052
	0.052	0.049	0.049	0.049	0.048	0.047	0.048	0.048	0.050
	0.057	0.053	0.051	0.052	0.049	0.052	0.051	0.053	0.053
	0.056	0.053	0.052	0.052	0.051	0.050	0.051	0.050	0.048
	0.056	0.052	0.053	0.052	0.051	0.051	0.051	0.050	0.050
	0.054	0.053	0.052	0.052	0.053	0.052	0.051	0.050	0.052

Table 10. Size of t_P^{B*} with sample size $T = 200$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.049	0.051	0.053	0.052	0.053	0.053	0.053	0.054	0.057
0.20	0.048	0.052	0.050	0.050	0.049	0.052	0.053	0.054	0.057
0.30	0.050	0.050	0.051	0.052	0.051	0.052	0.052	0.052	0.052
0.40	0.051	0.052	0.050	0.051	0.052	0.052	0.052	0.053	0.052
0.50	0.051	0.049	0.050	0.048	0.048	0.047	0.048	0.050	0.050
0.60	0.051	0.052	0.049	0.049	0.048	0.052	0.052	0.050	0.051
0.70	0.053	0.050	0.051	0.050	0.051	0.051	0.050	0.052	0.051
0.80	0.050	0.048	0.051	0.048	0.049	0.048	0.048	0.048	0.047
0.90	0.053	0.050	0.050	0.051	0.052	0.050	0.051	0.048	0.049

Table 11. Size of t_P^{C*} with sample size $T = 100$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.056	0.055	0.055	0.055	0.055	0.055	0.056	0.056	0.063
0.20	0.055	0.056	0.054	0.054	0.054	0.056	0.058	0.059	0.064
0.30	0.052	0.053	0.051	0.053	0.053	0.053	0.053	0.055	0.057
0.40	0.052	0.051	0.050	0.051	0.051	0.052	0.053	0.052	0.054
0.50	0.050	0.049	0.050	0.051	0.049	0.049	0.050	0.048	0.051
0.60	0.055	0.057	0.053	0.053	0.053	0.051	0.053	0.053	0.052
0.70	0.059	0.057	0.055	0.055	0.054	0.053	0.055	0.054	0.054
0.80	0.064	0.061	0.058	0.056	0.056	0.055	0.055	0.057	0.055
0.90	0.064	0.058	0.057	0.057	0.059	0.059	0.058	0.056	0.058

Table 12. Size of t_P^{C*} with sample size $T = 200$, $\mu_0 = \gamma = 0$

	$\delta = \sigma_2/\sigma_1$								
τ^c	4.00	2.50	1.67	1.25	1.00	0.80	0.60	0.40	0.25
0.10	0.055	0.052	0.051	0.053	0.055	0.053	0.053	0.053	0.055
0.20	0.052	0.054	0.051	0.051	0.052	0.052	0.054	0.053	0.055
0.30	0.054	0.053	0.051	0.053	0.052	0.053	0.052	0.053	0.055
0.40	0.052	0.055	0.052	0.053	0.051	0.053	0.051	0.054	0.052
0.50	0.048	0.050	0.051	0.050	0.050	0.049	0.050	0.050	0.049
0.60	0.052	0.051	0.052	0.051	0.052	0.051	0.049	0.051	0.050
0.70	0.054	0.054	0.052	0.050	0.052	0.052	0.053	0.052	0.051
0.80	0.055	0.052	0.051	0.049	0.050	0.052	0.050	0.051	0.050
0.90	0.056	0.055	0.053	0.051	0.053	0.051	0.053	0.051	0.051

Table 13. Modified Perron Unit Root Tests for the Nelson-Plosser Data Series

Series	Sample Period	Form-of -Break	τ^c	k^*	$\hat{\sigma}_1^i$	$\hat{\sigma}_2^i$	$\hat{\sigma}_2^i/\hat{\sigma}_1^i$	t_P^{i*}
Real GNP	1909-1970	A	0.33	8	0.0618	0.0375	0.61	-5.14^{a}
Nominal GNP	1909-1970	A	0.33	8	0.0959	0.0450	0.47	-4.13^{c}
Real Per Capita GNP	1909-1970	A	0.33	7	0.0650	0.0438	0.67	−3.51
Industrial Production	1860-1970	A	0.63	8	0.0839	0.0784	0.93	-4.68^{b}
Employment	1890-1970	A	0.49	7	0.0265	0.0274	1.03	-4.13^{d}
GNP Deflator	1889-1970	A	0.49	5	0.0525	0.0290	0.55	−3.20
Consumer Prices	1860-1970	A	0.63	2	0.0401	0.0296	0.74	−2.03
Nominal Wages	1900-1970	A	0.41	7	0.0628	0.0355	0.57	−3.22
Money Stock	1889-1970	A	0.49	6	0.0431	0.0370	0.86	−3.62
Velocity	1869-1970	A	0.59	0	0.0606	0.0683	1.13	−3.10
Interest Rate	1900-1970	A	0.41	2	0.2584	0.2704	1.05	−0.59
Common Stock Prices	1871-1970	C	0.59	1	0.1253	0.1504	1.20	−3.41
Real Wages	1900-1970	C	0.41	8	0.0330	0.0268	0.81	−3.90

Note: The small letters in parenthesis that appear as superscript indicate the significance of these statistics. The letters ‘a’, ‘b’, ‘c’, and ‘d’ indicate significance with respect to the asymptotic critical values at the 1%, 2.5%, 5%, and 10% significance level respectively. The asymptotic critical values of t_P^{i*} were extrapolated from Table VI.B of Perron (1989) based on the break-fraction.

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