Applied Mathematical Sciences, Vol. 3, 2009, no. 28, 1375 - 1380

High Order Parallel Iterative Method for Diffusion Equations

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Abstract

In this paper, we consider the initial boundary value problems of 1D diffusion equations. Based on a high order absolutely stable implicit scheme, we first construct two saul'yev asymmetry iterative schemes, and then present a class of parallel alternating group explicit iterative method in two cases. By use of the iterative method, the computation in the whole domain can be divided into many sub domains. Convergence analysis for the iterative method is also given. At the end of the paper the results of numerical experiments are presented.

Mathematics Subject Classification: 65M06, 74S20

Keywords: diffusion equations, iterative method, parallel computation, finite difference

1 Introduction

In order to solve $AX = F$, many finite difference methods have been presented. Considering the stability and accuracy of explicit schemes and the computation difficulty of implicit schemes, it is necessary to construct methods with the advantages of explicit methods and implicit methods, that is, simple for computation and good stability. Based on the concept of domain splitting, D. J. Evans presented an AGE method in [1] originally. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and then the numerical solutions at the group of points can be denoted explicitly. Furthermore, by alternating use of asymmetry schemes at adherent grid points and different time levels, the AGE method can lead to the property of unconditional stability. But the original AGE method has only two order accurate for spatial step. The AGE method is soon applied to convection-diffusion equations and hyperbolic equations in [2,3]. In [4,5], several AGE methods are presented for two-point linear and non-linear boundary value problems. Based on the concept of AGE method, a class of domain splitting method of order two accurate for 1D equations is presented in [6]. We notice that almost all the AGE method have no more than four order accurate in spatial step.

In this paper we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T \\ u(x,0) = f(x), \\ u(0,t) = g_1(t),\ u(1,t) = g_2(t). \end{cases} \tag{1}$$

In section 2, we construct two saul'yev asymmetry iterative schemes based on an $O(\tau^3 + h^6)$ order unconditionally stable symmetry implicit scheme in [7], and then present a class of parallel AGE iterative method. Convergence analysis for the method is given in section 3. The results of numerical example are presented at last.

2 The Parallel AGE Iterative Method

The domain $\Omega : [0,1]\times[0,T]$ will be divided into $(m\times\xi)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{\xi}$. Grid points are denoted by (x_j, t_n), $x_j = jh(i = 0,\ 1,\cdots,m)$, $t_n = n\tau(n = 0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (1) is denoted by u_j^n, while the exact solution $u(x_j,\ t_n)$. Let r=$\frac{a\tau}{h^2}$.

We present the implicit finite difference scheme with absolute stability in [7] for solving (1) as below:

$$(-24r^3-6r^2+r)u_{j-1}^{n+1}+(48r^3+60r^2+28r+3)u_j^{n+1}+(-24r^3-6r^2+r)u_{j+1}^{n+1} = 48r^2u_{j-1}^n+6u_j^n$$

$$+48r^2u_{j+1}^n+(-24r^3+6r^2+r)u_{j-1}^{n-1}+(48r^3-60r^2+28r-3)u_j^{n-1}+(-24r^3+6r^2+r)u_{j+1}^{n-1} \tag{2}$$

Applying Taylor formula to the scheme at (x_j, t_n). Considering $\frac{\partial^k u}{\partial t^k} = a^k\frac{\partial^{2k} u}{\partial x^{2k}}$, we have the truncation error $O(\tau^3 + h^6)$.

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_{m-1}^n)^T$, $a_1 = \frac{1}{2}(48r^3 + 60r^2 + 28r + 3)$, $a_2 = \frac{1}{2}(-24r^3 - 6r^2 + r)$, $a_3 = \frac{1}{2}(48r^3 - 60r^2 + 28r - 3)$, $a_4 = \frac{1}{2}(-24r^3 + 6r^2 + r)$, then we can denote (2) as $KU^{n+1} = \widetilde{E}^n$.

Here $\widetilde{E}^n = E_1U^n + E_2U^{n-1} + [(24r^3 + 6r^2 - r)u_0^{n+1} + 48r^2u_0^n + (-24r^3 +$

$6r^2+r)u_0^{n-1}, 0, \cdots, 0, (24r^3+6r^2-r)u_m^{n+1}+48r^2u_m^n+(-24r^3+6r^2+r)u_m^{n-1}]^T.$

$$K=\begin{pmatrix} 2a_1 & 2a_2 & & & \\ 2a_2 & 2a_1 & 2a_2 & & \\ & ... & ... & ... & \\ & & 2a_2 & 2a_1 & 2a_2 \\ & & & 2a_2 & 2a_1 \end{pmatrix}$$

$$E_1=\begin{pmatrix} 6 & 48r^2 & & & \\ 48r^2 & 6 & 48r^2 & & \\ & ... & ... & ... & \\ & & 48r^2 & 6 & 48r^2 \\ & & & 48r^2 & 6 \end{pmatrix}, \; E_2=\begin{pmatrix} 2a_3 & 2a_4 & & & \\ 2a_4 & 2a_3 & 2a_4 & & \\ & ... & ... & ... & \\ & & 2a_4 & 2a_3 & 2a_4 \\ & & & 2a_4 & 2a_3 \end{pmatrix}$$

K, E_1, E_2 are all $(m-1)\times(m-1)$ matrices.

Let k denotes the iterative number, $U^{n(k)}=(u_1^{n(k)}, u_2^{n(k)}, \cdots, u_{m-1}^{n(k)})^T$. Then based on (2) we present two saul'yev asymmetry iterative schemes as follows:

$$a_1u_j^{n+1(k+1)}+2a_2u_{j+1}^{n+1(k+1)}=-2a_2u_{j-1}^{n+1(k)}-a_1u_j^{n+1(k)}+\widetilde{E}^n \tag{3}$$

$$2a_2u_{j+1}^{n+1(k+1)}+a_1u_j^{n+1(k+1)}=-a_1u_j^{n+1(k)}-2a_2u_{j-1}^{n+1(k)}+\widetilde{E}^n \tag{4}$$

If we apply (3)-(4) to $(j,n+1),(j+1,n+1)$, it follows $u_j^{n+1(k+1)}, u_{j+1}^{n+1(k+1)}$ can be solved explicitly as one "two point group" as below:

$$\begin{pmatrix} U_j^{n+1(k+1)} \\ U_{j+1}^{n+1(k+1)} \end{pmatrix}=\begin{pmatrix} a_1 & 2a_2 \\ 2a_2 & a_1 \end{pmatrix}^{-1}\begin{pmatrix} -2a_2U_{j-1}^{n+1(k)}-a_1U_j^{n+1(k)}+\widetilde{E}^n \\ -a_1U_j^{n+1(k)}-2a_2U_{j-1}^{n+1(k)}+\widetilde{E}^n \end{pmatrix} \tag{5}$$

Let $U^{n+1(k)}=(u_1^{n+1(k)}, u_2^{n+1(k)}, \cdots, u_{m-1}^{n+1(k)})^T$, we construct the AGE iterative method in two cases as follows:

Case 1: Let $m-1=2s$, s is an integer. First in order to get the solution of $U^{n+1(k+\frac{1}{2})}$ with $U^{n+1(k)}$ known, we divide all the $(m-1)$ inner grid points into $\dfrac{(m-1)}{2}$ groups. Two grid points are included in each group, named $(j,n),(j,n+1)$, and (5) is applied to get the solution of $u_j^{n+1(k+1)}, u_{j+1}^{n+1(k+1)}$.

Second in order to get the solution of $U^{n+1(k+1)}$ with $U^{n+1(k+\frac{1}{2})}$ known, we divide all the grid points into $\dfrac{(m+1)}{2}$ groups. (4) and (3) are used to solve $u_1^{n+1(k+1)}$, $u_{m-1}^{n+1(k+1)}$ respectively. (5) is used to solve $(u_j^{n+1(k+1)},\ u_{j+1}^{n+1(k+1)})$, $j=2p(p=1,2,\cdots,\frac{m-3}{2})$.

By alternating use of the asymmetry schemes (3)-(4) the computation in the whole domain can be divided into many sub-domains. So the method has the obvious property of parallelism.

We denote the AGE iterative method I as follows:

$$\left\{\begin{array}{l}(\theta I+H_1)U^{n+1(k+\frac{1}{2})}=(\theta I-H_2)U^{n+1(k)}+\widetilde{E}^n\\(\theta I+H_2)U^{n+1(k+1)}=(\theta I-H_1)U^{n+1(k+\frac{1}{2})}+\widetilde{E}^n\end{array}\right. \quad k=0,1\cdots \tag{6}$$

Here θ is an iterative parameter.

$H_1=diag(H_{11},H_{11},\cdots,H_{11},H_{11})_{(m-1)\times(m-1)},\ H_2=diag(a_1,H_{11},\cdots,H_{11},a_1)_{(m-1)\times(m-1)}$

$$H_{11}=\begin{pmatrix}a_1 & 2a_2\\ 2a_2 & a_1\end{pmatrix}$$

Case 2: Let $m-1=2s+1$, s is an integer. First in order to get the solution of $U^{n+1(k+\frac{1}{2})}$ with $U^{n+1(k)}$ known, we divide all the $(m-1)$ inner grid points into $s+1$ groups. (3) is used to solve $u^{k+1}_{m-1,j+1}$, while (5) is used to solve $(u_j^{n+1(k+1)},\ u_{j+1}^{n+1(k+1)}),\ j=2p-1(p=1,2,\cdots,\frac{m-2}{2})$.

Second in order to get the solution of $U^{n+1(k+1)}$ with $U^{n+1(k+\frac{1}{2})}$ known, we still divide all the inner grid points into $s+1$ groups. (4) is used to solve $u^{k+1}_{1,j+1}$, while (5) is used to solve $(u_j^{n+1(k+1)},\ u_{j+1}^{n+1(k+1)}),\ j=2p(p=1,2,\cdots,\frac{m-2}{2})$.

Then we obtain the alternating group iterative method II :

$$\left\{\begin{array}{l}(\theta I+\widetilde{H}_1)U^{n+1(k+\frac{1}{2})}=(\theta I-\widetilde{H}_2)U^{n+1(k)}+\widetilde{E}^n\\(\theta I+\widetilde{H}_2)U^{n+1(k+1)}=(\theta I-\widetilde{H}_1)U^{n+1(k+\frac{1}{2})}+\widetilde{E}^n\end{array}\right. \quad k=0,1\cdots \tag{7}$$

$\widetilde{H}_1=diag(H_{11},H_{11},\cdots,H_{11},a_1)_{(m-1)\times(m-1)},\ H_2=diag(a_1,H_{11},\cdots,H_{11},H_{11})_{(m-1)\times(m-1)}$

3 Convergence Analysis For The AGE Method

Lemma 1 Let $\theta>0$, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\left\{\begin{array}{l}\|(\theta I+G)^{-1}\|_2\le\theta^{-1}\\ \|(\theta I-G)(\theta I+G)^{-1}\|_2\le 1\end{array}\right. \tag{8}$$

From the construction of the matrices we can see that $H_1,\ H_2,\ (H_1+H_1^T),\ (H_2+H_2^T)$ are all nonnegative matrices. Then we have

$$\|(\theta I-H_1)(\theta I+H_1)^{-1}\|_2\le 1,\ \|(\theta I-H_2)(\theta I+H_2)^{-1}\|_2\le 1$$

From (6), we can obtain $U^{n+1(k+1)}=HU^{n+1(k)}+(\theta I+H_2)^{-1}[(\theta I-H_1)(\theta I+H_1)^{-1}\widetilde{E}^n+\widetilde{E}^n]$. Here $H=(\theta I+H_2)^{-1}(\theta I-H_1)(\theta I+H_1)^{-1}(\theta I-H_2)$ is the growth matrix.

Let $\widetilde{H} = (\theta I + H_2)H(\theta I + H_2)^{-1} = (\theta I - H_1)(\theta I + H_1)^{-1}(\theta I - H_2)(\theta I + H_2)^{-1}$, then $\rho(H) = \rho(\widetilde{H}) \leq \|\widetilde{H}\|_2 \leq 1$, which shows the AGEI method given by (6) is convergent. Then we have:

Theorem 1 The AGE iterative method I given by (6) is convergent. Analogously we have:

Theorem 2 The AGE iterative method II given by (7) is also convergent.

4 Numerical Experiments

We consider the following example:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 \leq x \leq 1,\ 0 \leq t \leq T \\ u(x,0) = sin\pi x, \\ u(0,t) = 0,\ u(1,t) = 0. \end{cases} \quad (9)$$

The exact solution for the problem is $u(x,t) = e^{-\pi^2 t} sin\pi x$. Let $||E_1||_\infty$ denote maximum absolute error, while $||E_2||_\infty$ denote maximum relevant error. $||E_1||_\infty = |u_i^n - u(x_i, t_n)|$, $||E_2||_\infty = 100 \times |u_i^n - u(x_i, t_n)/u(x_i, t_n)|$. Let $\theta = 1$. In order to verify the AGEI method, we use the iterative error 1×10^{-10} to control the process of iterativeness, and the results of numerical experiments are listed in the following tables:

Table 1: The numerical results at $m = 17, \tau = 10^{-4}$

	$t = 100\tau$	$t = 500\tau$	$t = 1000\tau$	$t = 5000\tau$
$\|\|E_1\|\|_\infty$	8.909×10^{-4}	8.072×10^{-4}	3.671×10^{-4}	7.926×10^{-6}
$\|\|E_2\|\|_\infty$	9.875×10^{-2}	9.876×10^{-2}	9.891×10^{-2}	11.06×10^{-2}
average iterative times	6.93	6.965	6.774	5.769

Table 2: The numerical results at $m = 17, \tau = 5 \times 10^{-5}$

	$t = 100\tau$	$t = 500\tau$	$t = 1000\tau$	$t = 5000\tau$
$\|\|E_1\|\|_\infty$	4.679×10^{-4}	3.842×10^{-4}	3.003×10^{-4}	4.366×10^{-5}
$\|\|E_2\|\|_\infty$	4.936×10^{-2}	4.938×10^{-2}	4.941×10^{-2}	5.171×10^{-2}
average iterative times	5.94	5.988	5.994	5.324

5 Conclusions

From the results of table 1-2 we can see that the numerical solution by the iterative method presented in this paper can fast converge to the exact solution with high accurate. On the other hand, considering the whole domain can be divided into many sub-domains in computation, the AGE iterative method is suitable for parallel computation obviously.

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Received: November, 2008