Applied Mathematical Sciences, Vol. 3, 2009, no. 28, 1381 - 1385

AGE Method For Hyperbolic Equations

Qinghua Feng

School of Science, Shandong University of Technology
Zibo, Shandong, 255049, People's Republic of China
fengqinghua@sdut.edu.cn

Abstract

In this paper, we present a four order unconditionally stable implicit scheme for hyperbolic equations. Based on the scheme a class of parallel alternating group explicit (AGE) iterative method is derived, and convergence analysis is given. Results of numerical experiments show the iterative method can fast converge to the exact solution.

Mathematics Subject Classification: 65M06, 74S20

Keywords: hyperbolic equations, parallel computation, iterative method, alternating group, finite difference

1 Introduction

With the development of parallel computer many scientists payed much attention to the finite difference methods with the property of parallelism. D. J. Evans presented an AGE method in [1] originally. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and then the numerical solutions at the group of points can be denoted explicitly. Furthermore, by alternating use of asymmetry schemes at adherent grid points and different time levels, the AGE method can lead to the property of unconditional stability. But the original AGE method has only two order accurate for spatial step. The AGE method is soon applied to convection-diffusion equations and hyperbolic equations in [2,3]. In [4-5], Several AGE methods are given for two-point linear and non-linear boundary value problems. N. Bildik applied the concept of AGE method to poisson equations originally in [6], but the method has also only two order accurate. To our knowledge AGE methods for hyperbolic equations have scarcely been presented.

In this paper, we will consider the initial boundary value problem of 1D hyperbolic equations, and organize the rest of this paper as follows:

In section 2, we present a four order accurate unconditionally stable implicit scheme for hyperbolic equations. Based on the scheme we present two saul'yev asymmetry schemes, and construct a class of parallel AGE iterative method. Convergence analysis for the AGE iterative methods is given in section 3. Comparison between the method and the original AGE method in [3] is presented at the end of the paper.

2 The Four Order AGE Iterative Method

We consider the following initial boundary value problem of 1D hyperbolic equations:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T \\ u(x,0) = f(x), 0 \le x \le 1 \\ \dfrac{\partial u(x,0)}{\partial t} = \phi(x), 0 \le x \le 1 \\ u(0,t) = e_1(t),\ u(1,t) = e_2(t). \end{cases} \tag{1}$$

The domain $\Omega : [0,1] \times [0,T]$ will be divided into $(m \times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, t_n) or (i, n), $x_i = ih(i = 0,\ 1, \cdots, m),\ t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. Let $r = a^2 \frac{\tau^2}{h^2}$.

We present an absolutely stable implicit finite difference scheme with accurate of order $O(\tau^2 + h^4)$ for solving (1) as below:

$$(1-6r)u_{i-1}^{n+1} + (10+12r)u_i^{n+1} + (1-6r)u_{i+1}^{n+1} =$$

$$2u_{i-1}^n + 20u_i^n + 2u_{i+1}^n - (1-6r)u_{i-1}^{n-1} - (10+12r)u_i^{n-1} - (1-6r)u_{i+1}^{n-1} \tag{2}$$

Let $U^n = (u_1^n, u_2^n, \cdots, u_{m-1}^n)^T$, then we rewrite (3) as: $KU^{n+1} = E^n$.

Here $E^n = E_1 U^n + E_2 U^{n-1} + E_3$.

$E_3 = [-2u_0^n - (1-6r)u_0^{n-1} - (1-6r)u_0^{n+1}, 0, \cdots, 0, -2u_m^n - (1-6r)u_m^{n-1} - (1-6r)u_m^{n+1}]^T$.

$$K = \begin{pmatrix} 10+12r & 1-6r & & & \\ 1-6r & 10+12r & 1-6r & & \\ & \dots & \dots & \dots & \\ & & 1-6r & 10+12r & 1-6r \\ & & & 1-6r & 10+12r \end{pmatrix},\ E_1 = \begin{pmatrix} 20 & 2 & & & \\ 2 & 20 & 2 & & \\ & \dots & \dots & \dots & \\ & & 2 & 20 & 2 \\ & & & 2 & 20 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} -10-12r & -1+6r & & & \\ -1+6r & -10-12r & -1+6r & & \\ & \dots & \dots & \dots & \\ & & -1+6r & -10-12r & -1+6r \\ & & & -1+6r & -10-12r \end{pmatrix}$$

K, E_1, E_2 are all $(m-1)\times(m-1)$ matrices.

In order to solve U^{n+1} with U^n and U^{n-1} known, we will try to construct an AGE iterative method so as to avoid solving an implicit equation set.

We present two saul'yev asymmetry iterative schemes as follows:

$$(\rho+5+6r)u_i^{n+1(k+\frac{1}{2})}+(1-6r)u_{i+1}^{n+1(k+\frac{1}{2})} = -(1-6r)u_{i-1}^{n+1(k)}+\rho-(5+6r)u_i^{n+1(k)}+E_i^n \tag{3}$$

$$(1-6r)u_{i-1}^{n+1(k+\frac{1}{2})}+(\rho+5+6r)u_i^{n+1(k+\frac{1}{2})} = \rho-(5+6r)u_i^{n+1(k)}-(1-6r)u_{i+1}^{n+1(k)}+E_i^n \tag{4}$$

Here E_i^n is known. Based on the two schemes, we construct three basic computation groups:

"$\kappa 1$"group: two grid points are involved, and the following two iterative schemes are used respectively. Let $\widetilde{U}_i^n = (u_i^n, u_{i+1}^n)^T$, $\widetilde{U}_{i(k)}^n = (u_i^{n(k)}, u_{i+1}^{n(k)})^T$. Then the numerical solution of $\widetilde{U}_{i(k+1)}^{n+1}$ at grid nodes (i,n+1), (i+1,n+1) can be obtained explicitly as below:

$$\widetilde{U}_{i(k+1)}^{n+1} = \begin{pmatrix} 5+6r & 1-6r \\ 1-6r & 5+6r \end{pmatrix}^{-1} \begin{pmatrix} \rho-5-6r & -1+6r \\ -1+6r & \rho-5-6r \end{pmatrix} \widetilde{U}_{i(k)}^{n+1} + E_i^n \tag{5}$$

"$\kappa 2$"group: one inner point (m-1,n+1) is involved, and (3) is used.

"$\kappa 3$"group: one inner point (1,n+1) is involved, and (4) is used.

Let $U_k^{n+1} = (u_1^{n+1(k)}, u_2^{n+1(k)}, \cdots, u_{m-1}^{n+1(k)})^T$, then the alternating group iterative method will be constructed as follows:

Let $m-1=2s$, s is an integer. First in order to get the solution of $U_{k+\frac{1}{2}}^{n+1}$ with U_k^{n+1} known, we divide all the $(m-1)$ inner grid points into s groups. Two grid points are included in each group, named $(i,n),(i+1,n+1)$, and (5) is applied to get the solution of $u_i^{n+1(k+1)}, u_{i+1}^{n+1(k+1)}$.

Second in order to get the solution of U_{k+1}^{n+1} with $U_{k+\frac{1}{2}}^{n+1}$ known, we divide all the grid points into $s+1$ groups. (4) and (3) are used to solve $u_1^{n+1(k+1)}$, $u_{m-1}^{n+1(k+1)}$ respectively. (5) is used to solve $(u_i^{n+1(k+1)},\ u_{i+1}^{n+1(k+1)})$, $i = 2p(p=1,2,\cdots,\frac{m-3}{2})$.

By alternating use of the asymmetry schemes (3)-(4) the computation in the whole domain can be divided into many sub-domains. So the method has the obvious property of parallelism.

We denote the AGE iterative method I as follows:

$$\begin{cases} (\rho I + H_1)U_{k+\frac{1}{2}}^{n+1} = (\rho I - H_2)U_k^{n+1} + E^n \\ (\rho I + H_2)U_{k+1}^{n+1} = (\rho I - H_1)U_{k+\frac{1}{2}}^{n+1} + E^n \end{cases} \qquad k=0,1,\cdots \tag{6}$$

Here E^n, k is the iterative number, ρ is the iterative parameter.

$H_1 = diag(H_{11},\cdots,H_{11})_{(m-1)\times(m-1)},\ H_2 = diag(H_{21},H_{11},\cdots,H_{11},H_{21})_{(m-1)\times(m-1)}$

$$H_{11}=\begin{pmatrix} 5+6r & 1-6r \\ 1-6r & 5+6r \end{pmatrix},\ H_{21}=5+6r.$$

3 Convergence Analysis

Lemma 1 Let $\theta > 0$, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\begin{cases} \|(\theta I+G)^{-1}\|_2 \le \theta^{-1} \\ \|(\theta I-G)(\theta I+G)^{-1}\|_2 \le 1 \end{cases} \tag{8}$$

We can see that $H_1,\ H_2,\ (H_1+H_1^T),\ (H_2+H_2^T)$ are all nonnegative matrices. Then we have $\|(\theta I-H_1)(\theta I+H_1)^{-1}\|_2 \le 1,\ \|(\theta I-H_2)(\theta I+H_2)^{-1}\|_2 \le 1$. From (6), we can obtain $U_{k+1}^{n+1}=HU_k^{n+1}+(\rho I+H_2)^{-1}[(\rho I-H_1)(\rho I+H_1)^{-1}E^n+E^n]$. here $H=(\rho I+H_2)^{-1}(\rho I-H_1)(\rho I+H_1)^{-1}(\rho I-H_2)$ is the growth matrix.

Let $\widetilde{H}=(\rho I+H_2)H(\rho I+H_2)^{-1}=(\rho I-H_1)(\rho I+H_1)^{-1}(\rho I-H_2)(\rho I+H_2)^{-1}$, then $\rho(H)=\rho(\widetilde{H})\le\|\widetilde{H}\|_2\le 1$, which shows the AGEI method given by (6) is convergent. Then we have:

Theorem 1 The alternating group iterative method I is convergent.

4 Numerical Experiments

We consider the following problems:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2},\ 0\le x\le 1,\ 0\le t\le T \\ u(x,0)=0, \\ \dfrac{\partial u(x,0)}{\partial t}=sin(\pi x), 0\le x\le 1 \\ u(0,t)=0,\ u(1,t)=0. \end{cases} \tag{9}$$

The exact solution of (9) is denoted as below:

$$u(x,t)=sin\pi t sin(\pi x)$$

Let $||E_1||_\infty = max|u_i^n-u(x_i,t_n)|,\ ||E_2||_\infty=100\times max|(u_i^n-u(x_i,t_n))/u(x_i,t_n)|,\ i=1,2,\cdots,m-1$. We use the iterative error 1×10^{-10} to control the process of iterativeness. Then we compare the numerical results of the presented AGE iterative method in this paper with the method from [3] and the results from the implicit scheme (3) in table 1 . Let t_1/t_2 denote the ratio of running time between the AGE method and the implicit scheme (3).

Table 1: Results of comparison at $m = 17, \tau = 10^{-3},\ \rho = 1$

	$t = 100\tau$	$t = 200\tau$	$t = 500\tau$
average iterative times	50.49	50.54	48.68
$\|\|E_1\|\|_\infty$	2.978×10^{-4}	4.683×10^{-4}	2.475×10^{-5}
$\|\|E_1\|\|_\infty$[3]	3.104×10^{-2}	1.243×10^{-2}	7.638×10^{-3}
$\|\|E_2\|\|_\infty$	9.677×10^{-2}	4.336×10^{-2}	2.527×10^{-3}
$\|\|E_2\|\|_\infty$[3]	4.258	9.491×10^{-1}	6.573×10^{-1}
t_1/t_2	0.246	0.253	0.261

From the results of table 1 we can see that the numerical solution for the AGE method can fast converge to the exact solution, and is of higher accuracy than the original AGE method in [3], which accords to convergence analysis in section 5. Furthermore, for its intrinsic parallelism, the AGE method can shorten the running computing time in comparison with the fully implicit scheme (2).

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Received: November, 2008