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# A Remark on Hadamard's Inequality

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**Abstract.** In this paper we establish some inequalities by using a fairly elementary analysis. As corollary we obtain the well known Hadamard's inequality.

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## 1. Introduction

The following inequality, see for example [1-7],

$$\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)dx \leq \frac{f(x_1) + f(x_2)}{2},$$

which holds for convex function $f : (a, b) \to \mathbb{R}$ and $a < x_1 < x_2 < b$, is known in the literature as Hadamard's inequality. This inequality has been generalized and applied in various directions, see for example [1-3] and [6]. The aim of this paper is to establish some inequalities which generalize Hadamard's inequality and which are variants of Jensen's inequality for convex functions.

We shall use the following notation

$$x_\lambda = \lambda x_1 + (1 - \lambda)x_2,$$

$$J(f, x_1, x_2, \lambda) = \lambda f(x_1) + (1 - \lambda)f(x_2) - f(x_\lambda),$$

$$J^\star(f, x_1, x_2) = \sup_{\lambda \in [0,1]} J(f, x_1, x_2, \lambda),$$

$$H(f, x_1, x_2) = \frac{f(x_1) + f(x_2)}{2} - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x)dx, x_1 \neq x_2,$$

$$F(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, x_1 \neq x_2.$$

We need the following Lemma, see for example [1], [5] and [7] which deals with the simple characterization of convex functions.

**Lemma 1.** *The following statements are equivalent for a function* $f : (a, b) \to \mathbb{R}$.

$$J(f, x_1, x_2, \lambda) \geq 0 \text{ for all } a < x_1 < x_2 < b, \lambda \in [0, 1], \tag{1}$$

$$F(x_1, x) \leq F(x, x_2) \text{ for all } a < x_1 < x < x_2 < b, \tag{2}$$

$$f \text{ is continuous and } H(f, x_1, x_2) \geq 0 \text{ for all } a < x_1 < x_2 < b. \tag{3}$$

## 2. The results

Our main result is given in the following theorem.

**Theorem 1.** *Let* $f : (a, b) \to \mathbb{R}$ *be a convex function and let the derivative* $f'$ *exists, then*

$$\frac{J^\star(f, x_1, x_2)}{2} \leq H(f, x_1, x_2) \leq J^\star(f, x_1, x_2), \tag{4}$$

*for all* $a < x_1 < x_2 < b$.

*Proof.* Let $a < x_1 < x_2 < b$ and $\lambda \in [0, 1]$. Since $f$ is convex function we have

$$J(f, x_1, x_\lambda, \mu) \geq 0 \text{ for all } \mu \in [0, 1].$$

From Lemma 1. we obtain

$$F(x_1, x) \leq F(x, x_\lambda), \text{ for all } x_1 < x < x_\lambda.$$

So

$$f(x) \leq \frac{x_\lambda - x}{x_\lambda - x_1} \cdot f(x_1) + \frac{x - x_1}{x_\lambda - x_1} \cdot f(x_\lambda) \text{ for all } x_1 < x < x_\lambda. \tag{5}$$

Similarly, since

$$J(f, x_\lambda, x_2, \mu) \geq 0 \text{ for all } \mu \in [0, 1],$$

from Lemma 1. we have

$$f(x) \leq \frac{x_2 - x}{x_2 - x_\lambda} \cdot f(x_\lambda) + \frac{x - x_\lambda}{x_2 - x_\lambda} \cdot f(x_2) \text{ for all } x_\lambda < x < x_2. \tag{6}$$

Since

$$\int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_\lambda} f(x)dx + \int_{x_\lambda}^{x_2} f(x)dx,$$

using (5) and (6) we observe

$$\int_{x_1}^{x_2} f(x)dx \leq \frac{x_\lambda - x_1}{2} \cdot f(x_\lambda) + \frac{x_\lambda - x_1}{2} \cdot f(x_1) +$$

$$\frac{x_2 - x_\lambda}{2} \cdot f(x_2) + \frac{x_2 - x_\lambda}{2} \cdot f(x_\lambda).$$

So, we have

$$\int_{x_1}^{x_2} f(x)dx \leq \frac{x_2 - x_1}{2}[f(x_\lambda) + (1-\lambda)f(x_1) + \lambda f(x_2)],$$

$$\int_{x_1}^{x_2} f(x)dx \leq \frac{x_2 - x_1}{2}[f(x_1) + f(x_2) - J(f, x_1, x_2, \lambda)],$$

and

$$\frac{J(f, x_1, x_2, \lambda)}{2} \leq H(f, x_1, x_2).$$

Now we prove the right half of inequality (4). We prove that there exists $\lambda_c \in [0,1]$, such that

$$H(f, x_1, x_2) \leq J(f, x_1, x_2, \lambda_c).$$

Since derivative the $f'$ exists from Lagrange theorem we obtain that there exists $c \in (x_1, x_2)$, such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

So, exists $\lambda_c \in [0,1]$ such that

$$c = \lambda_c x_1 + (1 - \lambda_c)x_2. \tag{7}$$

We prove that

$$H(f, x_1, x_2) \leq J(f, x_1, x_2, \lambda_c).$$

Since $f$ is convex function, we have

$$f(x) \geq f(c) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - c) \text{ for all } a < x_1 < x < x_2 < b. \tag{8}$$

Using (8) we obtain

$$\int_{x_1}^{x_2} f(x)dx \geq f(c)(x_2 - x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot \frac{(x_2 - c)^2}{2} -$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot \frac{(x_1 - c)^2}{2}.$$

Now using (7) we conclusion

$$\int_{x_1}^{x_2} f(x)dx \geq \frac{x_2 - x_1}{2}[f(x_1) + f(x_2) + 2(f(c) - \lambda_c f(x_1) - (1 - \lambda_c)f(x_2))],$$

so,

$$H(f, x_1, x_2) \leq J(f, x_1, x_2, \lambda_c).$$

□

**Remark 1.** For the proof of the left side of inequality (4) it is enough to assume that function $f$ is convex. In this case as corollary we obtain Hadamard's inequality. As immediate corollary, we obtain the following inequality

$$|J(f, x_1, x_2, \lambda) - H(f, x_1, x_2)| \leq H(f, x_1, x_2) \text{ for all } a < x_1 < x_2 < b, \lambda \in [0, 1].$$

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