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Alternating Group Iterative Method for Diffusion Equations

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Abstract

In this paper, a class of alternating group explicit iterative parallel method(AGI) is derived for solving diffusion equations based on a high order implicit scheme. Also the convergence analysis and the stability analysis are given. In order to verify the results, several numerical examples are presented in the end of the paper.

Mathematics Subject Classification: 65M06, 74S20

Keywords: iterative method, parallel computation, alternating group, diffusion equations

1 Introduction

we consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T \\ u(x,0) = f(x), \\ u(0,t) = g_1(t),\ u(1,t) = g_2(t). \end{cases} \tag{1.1}$$

With the development of parallel computer, researches on parallel numerical methods are getting more and more popular. Considering the stability and accuracy of common explicit schemes and the computation difficulty of most implicit schemes, it is a great task to present parallel numerical methods with absolute stability. A class of alternating group method (AGE) for paraboic equations is presented in [1,2]. The AGE method is widely cared for it is simple for computing, unconditionally stable, and has the property of parallelism. Based on the AGE method, many alternating group methods have been presented such as in[3-7], also a class of domain splitting finite difference method is presented in[8]. Most of the methods inherit the advantages of the AGE method, but almost all the methods have $O(h^2)$ accuracy for spatial step in the case of using six grid points. On the other hand we notice that researches on alternating group iterative methods are scarcely presented.

The construction of this paper is as follows: In section 2, we present a $O(\tau^2 + h^4)$ order unconditionally stable symmetry six-point implicit scheme for solving (1.1) at first. Based on the scheme two alternating group explicit iterative methods(AGI1 and AGI2) are constructed. In section 3, convergence analysis and stability analysis are given. In section 4, results of several numerical examples are presented.

2 The Alternating Group Iterative Method(AGI)

The domain $\Omega : (0,1) \times (0,T)$ will be divided into $(m \times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, t_n), $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$.

We present an implicit finite difference scheme with parameters for solving (1.1) as below:

$$\theta_1\frac{u_{j-1}^{n+1}-u_{j-1}^{n}}{\tau}+\theta_2\frac{u_{j}^{n+1}-u_{j}^{n}}{\tau}+\theta_1\frac{u_{j+1}^{n+1}-u_{j+1}^{n}}{\tau}=\frac{a}{2}[(\frac{u_{i+1}^{n+1}-2u_{i}^{n+1}+u_{i-1}^{n+1}}{h^2}+(\frac{u_{i+1}^{n}-2u_{i}^{n}+u_{i-1}^{n}}{h^2})] \tag{2.1}$$

Applying Taylor's formula to the scheme at (x_i, t_n). Considering $\frac{\partial^k u}{\partial t^k} = a^k\frac{\partial^{2k} u}{\partial t^{2k}}$, then we have the truncation error:

$$(\theta_1+\theta_2+\theta_3-1)a\frac{\partial^2 u}{\partial x^2}+(-\theta_1+\theta_3)ah\frac{\partial^3 u}{\partial x^3}-\frac{1}{2}(\theta_1+\theta_2+\theta_3-1)a^2\tau\frac{\partial^4 u}{\partial x^4}+\frac{1}{2}(\theta_1+\theta_3-\frac{1}{6})ah^2\frac{\partial^4 u}{\partial x^4}$$
$$+\frac{1}{6}(\theta_3-\theta_1)ah^3\frac{\partial^5 u}{\partial x^5}+\frac{1}{2}(\theta_1-\theta_3)a^2\tau h\frac{\partial^5 u}{\partial x^5}+\frac{1}{4}(-\theta_1-\theta_3+\frac{1}{6})a^2\tau h^2\frac{\partial^6 u}{\partial x^6}+\frac{1}{12}(\theta_1-\theta_3-\frac{1}{6})a^2\tau h^3\frac{\partial^7 u}{\partial x^7}$$
$$+O(\tau^2+h^4)$$

Let

$$\begin{cases} \theta_1+\theta_2+\theta_3-1=0 \\ -\theta_1+\theta_3=0 \\ \theta_1+\theta_3-\frac{1}{6}=0 \end{cases}$$

that is $\theta_1 = \theta_3 = \frac{1}{12},\ \theta_2 = \frac{5}{6}$. Then we can easily have that the truncation error of the scheme is $O(\tau^2 + h^4)$.

Let r$=\frac{\tau}{24h^2}$, then from (2.1) we have:

$$(1-6r)u_{i-1}^{n+1}+(10+12r)u_i^{n+1}+(1-6r)u_{i+1}^{n+1} = (1+6r)u_{i-1}^n+(10-12r)u_i^n+(1+6r)u_{i+1}^n \quad (2.2)$$

Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_{m-1}^n)^T$, then from (2.1) we can have $AU^{n+1} = F^n$. here

$$A = \begin{pmatrix} 10+12r & 1-6r & & & \\ 1-6r & 10+12r & 1-6r & & \\ & \cdots & \cdots & \cdots & \\ & & 1-6r & 10+12r & 1-6r \\ & & & 1-6r & 10+12r \end{pmatrix}_{(m-1)\times(m-1)}$$

$$F^n = (2I-A)U^n+[(1+6r)u_0^n-(1-6r)u_0^{n+1}, 0, \cdots, 0, (1+6r)u_m^n-(1-6r)u_m^{n+1}]^T$$

The alternating group iterative method will be constructed in two cases as follows:
(1)$m = 4k,\ k$ is an integer. Let $A = \frac{1}{2}(G_1 + G_2)$, here

$$G_1 = diag(G_{11}, \cdots, G_{11})_{(m-1)\times(m-1)},\ G_2 = diag(G_{21}, G_{11}, \cdots, G_{11}, G_{21})_{(m-1)\times(m-1)}$$

$$G_{11} = \begin{pmatrix} 10+12r & 1-6r & 0 & 0 \\ 0 & 10+12r & 2(1-6r) & 0 \\ 0 & 2(1-6r) & 10+12r & 1-6r \\ 0 & 0 & 1-6r & 10+12r \end{pmatrix},\ G_{21} = \begin{pmatrix} 10+12r & 1-6r \\ 1-6r & 10+12r \end{pmatrix}$$

Then the alternating group iterative method (AGI1) can be denoted as below:

$$\begin{cases} (\rho I + G_1)\tilde{u}^{n+1} = (\rho I - G_2)u^n + 2F^n \\ (\rho I + G_2)u^{n+1} = (\rho I - G_1)\tilde{u}^{n+1} + 2F^n \end{cases} \quad (2.3)$$

(2)$m = 4k+2,\ k$ is an integer. Let $A = \frac{1}{2}(\overline{G}_1 + \overline{G}_2)$, here

$$\overline{G}_1 = diag(G_{11}, \cdots, G_{11}, G_{21})_{(m-1)\times(m-1)},\ \overline{G}_2 = diag(G_{21}, G_{11}, \cdots, G_{11})_{(m-1)\times(m-1)}$$

Then the alternating group iterative method (AGI2) can be derived as below:

$$\begin{cases} (\rho I + \overline{G}_1)\tilde{u}^{n+1} = (\rho I - \overline{G}_2)u^n + 2F^n \\ (\rho I + \overline{G}_2)u^{n+1} = (\rho I - \overline{G}_1)\tilde{u}^{n+1} + 2F^n \end{cases} \quad (2.4)$$

3 Convergence And Stability Analysis

In order to verify the convergence of the AGI method, from [9] we have the following lemmas:
Lemma 1 Let θ >0, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\|(\theta I+G)^{-1}\|_2 \leq \theta^{-1} \tag{3.1}$$

Lemma 2 On the conditions of Lemma 1, we have:

$$\|(\theta I-G)(\theta I+G)^{-1}\|_2 \leq 1 \tag{3.2}$$

Theorem 1 The alternating group iterative method is convergent.

proof: From the construction of the matrixes we can see that $G_1,\ G_2,\ (G_1+G_1^T),\ (G_2+G_2^T)$ are all nonnegative matrixes. Then we have

$$\|(\rho I-G_1)(\rho I+G_1)^{-1}\|_2 \leq 1,\ \|(\rho I-G_2)(\rho I+G_2)^{-1}\|_2 \leq 1$$

From (2.2), we can obtain $U^{n+1}=GU^n+2(\rho I+G_2)^{-1}[(\rho I-G_1)(\rho I+G_1)^{-1}F^n+F^n]$. here $G=(\rho I+G_2)^{-1}(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)$ is growth matrix.

Let $\widetilde{G}=(\rho I+G_2)G(\rho I+G_2)^{-1}=(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)(\rho I+G_2)^{-1}$, then $\rho(G)=\rho(\widetilde{G})\leq\|\widetilde{G}\|_2\leq 1.$, which shows the AGI1 method given by (2.3) is convergent.

Analogously we have: The alternating group iterative method (AGI2) given by (2.4) is also convergent.

In order to analyze the stability of (2.2), we will use the Fourier method. Let $U^n=V^ne^{i\alpha x_j}$, then from (2.2) we have

$$V^{n+1}=V^n\frac{10-12r+(1+6r)cos(\alpha h)}{10+12r+(1-6r)cos(\alpha h)}. \tag{3.3}$$

Obviously we have $|\frac{10-12r+(1+6r)cos(\alpha h)}{10+12r+(1-6r)cos(\alpha h)}|\leq 1$. So we can get the following theorem:

Theorem 2 The scheme (2.2) is unconditionally stable.

4 Numerical Experiments

We consider the following initial boundary value problem of diffusion equations:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2},\ 0\leq x\leq 1,\ 0\leq t\leq T \\ u(x,0)=sin(\pi x), \\ u(0,t)=0,\ u(1,t)=0. \end{cases} \tag{4.1}$$

The exact solution for the problem is $u(x,t) = e^{-\pi^2 t} sin(2\pi x)$. Let A.E denote maximum absolute error, while P.E denote maximum relevant error. A.E=$|u_i^n - u(x_i,t_n)|$, P.E=$100 \times |u_i^n - u(x_i,t_n)/u(x_i,t_n)|$. We will use the iterative error 1×10^{-6} to control the process of iterativeness, and the results of numerical experiments is listed in the following two tables:

Table 1: The numerical results for the iterative method AGI1

	$m = 17, \tau = 10^{-3}, t = 100\tau$	$m = 17, \tau = 10^{-4}, t = 100\tau$
A.E	1.428×10^{-4}	2.071×10^{-4}
P.E	3.869×10^{-2}	2.345×10^{-2}
average iterative times	23.58	18

Table 2: The numerical results for the iterative method AGI1

	$m = 17, \tau = 10^{-4}, t = 1000\tau$	$m = 21, \tau = 10^{-3}, t = 100\tau$
A.E	1.404×10^{-3}	9.603×10^{-5}
P.E	3.795×10^{-1}	2.592×10^{-2}
average iterative times	16.681	24.32

Table 3: The numerical results for the iterative method AGI2

	$m = 19, \tau = 10^{-3}, t = 100\tau$	$m = 19, \tau = 10^{-4}, t = 100\tau$
A.E	1.153×10^{-4}	2.113×10^{-4}
P.E	3.111×10^{-2}	2.351×10^{-2}
average iterative times	24.05	18

Table 4: The numerical results for the iterative method AGI2

	$m = 19, \tau = 10^{-4}, t = 1000\tau$	$m = 23, \tau = 10^{-3}, t = 100\tau$
A.E	1.298×10^{-3}	6.900×10^{-5}
P.E	3.498×10^{-1}	1.889×10^{-2}
average iterative times	16.917	24.67

From table 1 and table 2 we can see that the numerical solution for the AGI methods (2.2) and (2.3) can converge to the exact solution fast, and we can get higher accuracy when the spatial step diminishes, which accords to the conclusion of convergence and error analysis. On the other hand, the AGI methods have the obvious property of parallelism.

5 Conclusions

In this paper, an universal alternating group iterative(AGI) method is derived by using a special implicit scheme of high accuracy, also convergence analysis and stability analysis are finished. The AGI method is convenient to use in solving large equation set, and is suitable for parallel computation. Furthermore, using the Tylor formula we can obtain another implicit scheme of higher accuracy than (2.1), and based on the scheme we can derive another alternating group iterative method by almost the same procession in section 2. The construction of the AGI method can also be applied to other partial differential equations.

References

[1] D. J. Evans , A. R. Abdullah, Group Explicit Method for Parabolic Equations [J]. Inter J Computer Math. 14(1983)73 105.

[2] D. J. Evans and A. R. Abdullah, A New Explicit Method for Diffusion-Convection Equation, Comp. Math. Appl. 11(1985)145-154.

[3] D. J. Evans, H. Bulut, The numerical solution of the telegraph equation by the alternating group explicit(AGE) method[J], Inter J Computer Math. 80(2003)1289-1297.

[4] J. Gao, G. He, An unconditionally stable parallel difference scheme for parabolic equations, Appl. Math. Comput. 135(2003)391-398.

[5] G. W. Yuan, L. J. Shen, Y.L.Zhou, Unconditional stability of parallel alternating difference schemes for semilinear parabolic systems, Appl. Math. Comput, 117(2001)267-283.

[6] J. f. Jin, B. l. Zhang, F. l. Zuo, Modified AGE methods for The Convection-Diffusion Equation, Communications in Numerical Methods in Engineering, 14(1998)65-75.

[7] S. Zhu, G. Yuan, W. Sun, Convergence and stability of explicit/implicit schemes for parabolic equations with discontinuous coeffients, Int. J. Numer. Anal. Model, 1(2004)131-145.

[8] R. Tavakoli, P. Davami, New stable group explicit finite difference method for solution of diffusion equation, Appl. Math. Comput, 181(2006)1379-1386.

[9] R. B. Kellogg, An alternating Direction Method for Operator Equations[J]. J Soc Indust Appl. Math(SIAM), 12(1964)848-854.

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