

# Sensitivity Analysis with Fuzzy Data in DEA

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## Abstract

One of the interesting topics in data envelopment analysis is the sensitivity analysis of the decision making units, under evaluation. In this paper we present a new method for sensitivity analysis of the decision making units with fuzzy data, and finally we illustrate the proposed method by a numerical example .

**Keywords:** DEA; Fuzzy DEA; Sensitivity analysis

## 1 Introduction

DEA (Data Envelopment Analysis) is a non-parametric technique for measuring the efficiency of DMUs (Decision Making Units) with common inputs and outputs [2,5]. viewpoint for each DMU because of taking a maximum ratio. During recent years, the issue of sensitivity and stability of data envelopment analysis results has been extensively studied. The first DEA sensitivity analysis paper by Charnes et al. [3] examined change in a single output. This was followed by a series of sensitivity analysis articles by Charnes and Neralic [4] in which sufficient conditions for preserving efficiency are determined. Another type of DEA sensitivity analysis is based on super-efficiency DEA approach in which the DMU under evaluation is not included in the reference set [1,11]. Charnes et al. [6,7] developed a super-efficiency DEA sensitivity analysis technique for the situation where simultaneous proportional change is assumed in all inputs and outputs for a specific DMU under consideration.

In traditional DEA models, it is assumed that all inputs and outputs are exactly known. But in real world, this assumption may not always be true.

On the other hand, in more general cases, the data for evaluation are stated by natural language, such as good or bad, to reflect the general situation. So some researchers have proposed DEA fuzzy models to evaluate DMUs with fuzzy data [9,10,12].

However, methods of sensitivity analysis proposed in DEA are not suitable

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for DMUs with fuzzy data. So in this paper we propose a new method for sensitivity analysis of such DMUs.

This paper consist of the following sections: In section 2, fuzzy numbers are introduced. Sensitivity analysis with fuzzy data is presented in section 3. An example with fuzzy data will be provided in section 4, and finally the conclusion is given in section 5.

## 2 Fuzzy numbers

A fuzzy set  $A$  in  $X$  is a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

$\mu_A(x)$  is called the membership function of  $x$  in  $A$ .

A fuzzy number  $M$  is a convex normalized fuzzy set  $M$  of real line  $R$  such that :

- 1) there exists exactly one  $x \in R$  with  $\mu_M(x) = 1$ .
- 2)  $\mu_M(x)$  is piecewise continuous.

The (crisp) set of element that belong to the fuzzy set  $A$  at least to the degree  $\alpha$ -cut set:

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}.$$

The lower and upper endpoints of any  $\alpha$ -cut set,  $A_\alpha$ , are represented by  $\underline{A}(\alpha)$  and  $\overline{A}(\alpha)$ , respectively.

**Definition 1.** Given two symmetric triangular fuzzy variables  $Z_1 = (z_1, w_1)$  and  $Z_2 = (z_2, w_2)$ , the relation  $z_1 \preceq z_2$  is defined by the following inequalities:

$$z_1 - (1 - h)w_1 \leq z_2 - (1 - h)w_2,$$

$$z_1 + (1 - h)w_1 \leq z_2 + (1 - h)w_2.$$

## 3 Sensitivity analysis with fuzzy data

Cooper et al. [8] proposed a method for the sensitivity analysis of efficient DMUs and inefficient DMUs. Consider the following models for the the sensitivity analysis of inefficient DMUs,

$$\begin{aligned}
& \max \quad \delta & (1) \\
& \text{s.t.} \\
& y_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n y_{rj} \lambda_j - \delta, \\
& x_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n x_{ij} \lambda_j + \delta, \\
& \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j = 1, \\
& \lambda_j \geq 0, & j = 1, \dots, n, \\
& \delta \geq 0.
\end{aligned}$$

Let us assume that we have a set of DMUs consisting of  $DMU_j$ ,  $j=1, \dots, n$ , with fuzzy input-output vectors  $(\tilde{x}_j, \tilde{y}_j)$ , in which  $\tilde{x}_j = (x_j, c_j) \in F(R)_{\geq 0}$  and  $\tilde{y}_j = (y_j, d_j) \in F(R)_{\geq 0}$  where  $F(R)_{\geq 0}$  is a family of all nonnegative fuzzy numbers.

So we extended model(1) to the fuzzy case as follows:

$$\begin{aligned}
& \max \quad \tilde{\delta} & (2) \\
& \text{s.t.} \\
& \tilde{y}_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \tilde{y}_{rj} \lambda_j - \tilde{\delta}, \\
& \tilde{x}_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \tilde{x}_{ij} \lambda_j + \tilde{\delta}, \\
& \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j = 1, \\
& \lambda_j \geq 0, & j = 1, \dots, n, \\
& \tilde{\delta} \in F(R^+).
\end{aligned}$$

Now we want to compute the stability radius for inefficient DMUs with fuzzy data. We compute the stability radius for different  $\alpha$ -levels.

Now, let us consider maximizing the fuzzy variable. By using definition [10], maximizing the fuzzy variable  $\delta = (\delta_0, x)$  can be restated as simultaneously maximizing  $\delta_0 - (1 - \alpha)x$  and  $\delta_0 + (1 - \alpha)x$ . A weighted function  $\lambda_1(\delta_0 - (1 - \alpha)x) + \lambda_2(\delta_0 + (1 - \alpha)x)$  is introduced where  $\lambda_1, \lambda_2$  are the weights of the left and right endpoints of the  $\alpha$ -level set of  $\delta$ , respectively, with  $\lambda_1 + \lambda_2 = 1$ . Taking  $\lambda_1 = 1$  is regarded as a pessimistic opinion of maximizing  $z$  because the worst situation is considered. In this paper,  $\lambda_1$  is taken as 1, that is,

$$\max \delta_0 - (1 - \alpha)x.$$

By using the replacement definition 1 for ranking fuzzy numbers and the definition of maximizing the fuzzy variable, the fuzzy optimization problem (2) can be transformed into the following L.P problem:

$$\begin{aligned} \max \quad & \delta_0 - (1 - \alpha)x & (3) \\ \text{s.t.} \quad & \\ & y_{ro} - (1 - \alpha)d_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j y_{rj} - (1 - \alpha)d_{rj} - \delta_o - (1 - \alpha)e_o, \\ & y_{ro} + (1 - \alpha)d_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j y_{rj} + (1 - \alpha)d_{rj} - \delta_o + (1 - \alpha)e_o, \\ & x_{io} - (1 - \alpha)c_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j x_{ij} - (1 - \alpha)c_{ij} + \delta_o - (1 - \alpha)e_o, \\ & x_{io} + (1 - \alpha)c_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j x_{ij} + (1 - \alpha)c_{ij} + \delta_o + (1 - \alpha)e_o, \\ & \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j = 1, \\ & \lambda_j \geq 0 & j = 1, \dots, n, \\ & \delta_o \geq 0 \quad e_o \geq 0. \end{aligned}$$

Now, we consider the method proposed by Cooper et al. [8] for inefficient DMUs and extend it to the fuzzy case, as follows:

$$\begin{aligned} \min \quad & \tilde{\delta} & (4) \\ \text{s.t.} \quad & \\ & \tilde{y}_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \tilde{y}_{rj} \lambda_j + \tilde{\delta}, \\ & \tilde{x}_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \tilde{x}_{ij} \lambda_j - \tilde{\delta}, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0, & j = 1, \dots, n, \\ & \tilde{\delta} \in F(R^+). \end{aligned}$$

We want to compute the instability radius for different  $\alpha$ -levels. Minimizing the fuzzy variable  $\delta = (\delta_0, x)$  can be restated as simultaneously minimizing  $\delta_0 - (1 - \alpha)x$  and  $\delta_0 + (1 - \alpha)x$ . A weighted function  $\lambda_1(\delta_0 - (1 -$

$\alpha)x) + \lambda_2(\delta_0 + (1 - \alpha)x)$  is introduced where  $\lambda_1 + \lambda_2 = 1$ . Taking  $\lambda_1 = 1$  is regarded as a pessimistic opinion of minimizing  $z$  because the worst situation is considered. In this paper  $\lambda_1$  is taken as 1, that is,  $\min \delta_0 - (1 - \alpha)x$ . So we have:

$$\begin{aligned}
 & \min \quad \delta_0 - (1 - \alpha)x & (5) \\
 & s.t. \\
 & y_{ro} - (1 - \alpha)d_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j y_{rj} - (1 - \alpha)d_{rj} + \delta_o - (1 - \alpha)e_o, \\
 & y_{ro} + (1 - \alpha)d_{ro} \leq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j y_{rj} + (1 - \alpha)d_{rj} + \delta_o + (1 - \alpha)e_o, \\
 & x_{io} - (1 - \alpha)c_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j x_{ij} - (1 - \alpha)c_{ij} - \delta_o - (1 - \alpha)e_o, \\
 & x_{io} + (1 - \alpha)c_{io} \geq \sum_{\substack{j=1 \\ j \neq 0 \\ n}}^n \lambda_j x_{ij} + (1 - \alpha)c_{ij} - \delta_o + (1 - \alpha)e_o, \\
 & \sum_{j=1}^n \lambda_j = 1, \\
 & \lambda_j \geq 0 \\
 & \delta_o \geq 0 \quad e_o \geq 0.
 \end{aligned}
 \qquad j = 1, \dots, n,$$

## 4 Numerical example

In this section, to illustrate the use of the methodology developed here, a numerical example is provided. In this example we have  $L(x) = R(x) = 1 - x$  for inputs, and outputs. At first, by the method proposed in [13], we evaluate  $DMU_A, \dots, DMU_E$  and identify the efficient and inefficient DMUs. The results of evaluation are presented in Table 1. Then, by method proposed in this paper, we compute radius of stability and unstability in different  $\alpha$ -levels for efficient and inefficient DMUs. The results of the evaluation are summarized in Table 2.

DMUs	Input 1	Input 2	output 1	output 2	efficiency
A	(4,0.5)	(2,0.2)	(5,0.5)	(4, 0.3)	1
B	(3,5)	(1.5, 0.1)	(2, 0.2)	(3.5,0.2)	1
C	(5,0.5)	(2.5,0.5)	(3, 0.3)	(5,1)	0.7535
D	(4,0.7)	(2,0.1)	(3,0.5)	(6.1)	1
E	(6,0.6)	(4,0.5)	(5,0.7)	(7.2)	1

Table 1. Results of the evaluation by the method in [13].

	0	0.2	0.4	0.6	0.8	1
A	2.75	2.4	2.05	1.7	1.35	1
B	3.3	2.84	2.38	1.92	1.46	1
C	4.6222	3.2422	0.3433	0.1307	0.1143	0.5
D	3.0923	2.5969	2.1015	1.6337	1.1853	0.73
E	3.5	3.1	2.7	2.3	1.9	1.5

Table 2. Results of the evaluation by our proposed method.

## 5 Conclusion.

In ordinary methods for sensitivity analysis in DEA, it is assumed that all data are exactly known, but in practice this assumption is not always true. So, in this paper, we proposed a new method for sensitivity analysis with fuzzy data.

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